

NONNEGATIVELY CURVED ALEXANDROV SPACES WITH SOULS OF CODIMENSION TWO

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ABSTRACT. In this paper, we study a complete noncompact nonnegatively curved Alexandrov space A with a soul S of codimension two. We establish some structural results under additional regularity assumptions. As an application, we conclude that in this case Sharafutdinov retraction, $\pi : A \rightarrow S$, is a submetry.

INTRODUCTION

We begin with the classical Soul Theorem of Cheeger-Gromoll ([10]) on complete noncompact Riemannian manifolds of nonnegative sectional curvature:

Theorem 0.1. *Let M be a complete noncompact Riemannian n -manifold with sectional curvature $\sec(M) \geq 0$. Then M contains a compact totally geodesic submanifold S (called a soul of M) such that M is diffeomorphic to the normal bundle of S .*

When $\sec(M) > 0$, Gromoll and Meyer ([15]) earlier showed that a soul is a point, and thus M is diffeomorphic to \mathbb{R}^n . Cheeger and Gromoll proposed the following so called Soul Conjecture: If a complete noncompact nonnegatively curved Riemannian manifold has strictly positive sectional curvature around a point, then a soul is a point.

In 1994, Perelman ([26]) proved the following theorem which implies the Soul Conjecture:

Theorem 0.2. *Let M be a complete noncompact Riemannian n -manifold with $\sec(M) \geq 0$, and let S be a soul. If $P : M \rightarrow S$ is a distance nonincreasing map, then the following properties hold:*

(0.2.1) *For any $x \in S$ and any unit vector v at x normal to S , $P(\exp_x(tv)) = x$, for all $t \geq 0$.*

(0.2.2) *Let $\gamma : [0, l] \rightarrow S$ be a geodesic, and let $V(s)$ denote the parallel vector field along $\gamma(s)$ with $V(0) = v$. Then $\sigma_s(t) = \exp_{\gamma(s)}(tV(s))$ are geodesics filling a flat totally geodesic strip ($t \geq 0$). If $\gamma([0, l])$ is minimal, then $\sigma_s(t_0)|_{[0, l]}$ are minimal for any fixed t_0 .*

(0.2.3) *P is a C^1 -Riemannian submersion such that the eigenvalue of the second*

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fundamental form of P -fibers are bounded above by the inverse of the injectivity radius of S (in the barrier sense).

Note that (0.2.2) implies that if S is not a point, then any point in M is on some flat totally geodesic strip, and thus the Soul Conjecture.

Note that it was shown independently by Cao-Shaw ([12]) and Wilking ([39]) that $P : M \rightarrow S$ is actually smooth (cf. [17]).

In this paper, we are concerned with analogue of Theorem 0.2 in Alexandrov geometry. An Alexandrov space is a complete length space on which Toponogov's triangle comparison holds. The study of Alexandrov spaces was initiated by Burago-Gromov-Perelman ([3]), partially motivated by the fact that the Gromov-Hausdorff limit of a sequence of Riemannian manifolds with sectional curvature bounded from below uniformly is an Alexandrov space which in general may have both geometric and topological singularities.

In [24, 6.3], Perelman extended Theorem 0.1 to Alexandrov spaces:

Theorem 0.3. *Let A be a complete noncompact nonnegatively curved Alexandrov space. Then there is a compact convex subset S without boundary and a distance nonincreasing deformation retraction $\pi : A \rightarrow S$.*

The map π is called the Sharafutdinov retraction of A . In further study following Theorem 0.3, there are two basic questions:

Open problem 0.4. (0.4.1) ([26]) Soul Conjecture: If a complete noncompact nonnegatively curved Alexandrov space has strictly positive curvature around a point, then a soul is a point.

(0.4.2) ([1, 18.5]) Is the Sharafutdinov retraction π a submetry? (A submetry is a map which preserves all r -balls, and thus submetry is a metric analogue of Riemannian submersion.)

An affirmative answer to (0.4.2) will easily imply (0.4.1) (see the proof of Corollary 0.6), but the converse may not be true.

(In the following context, when we say a minimal geodesic from one point to a subset, we always mean one whose length realizes the distance from the point to the subset.)

In [42, 2.1], Yamaguchi partially generalized (0.2.2):

Theorem 0.5. *Let C be a convex closed subset in $A \in \text{Alex}^n(0)$, with boundary $\partial C \neq \emptyset$, let $f = \text{dist}_{\partial C}$ and let $\gamma(t) \subset C$ ($t \in [0, b]$) be a minimal geodesic with $\gamma(0) = p$, $\gamma(b) = q$, such that $f(\gamma(t)) = \text{const}$. Then for any minimal geodesic γ_0 from p to ∂C , with $\angle(\gamma_0^+(0), \gamma^+(0)) = \frac{\pi}{2}$, there is a minimal geodesic γ_1 from q to ∂C , such that $\{\gamma, \gamma_0, \gamma_1\}$ bounds a flat totally geodesic rectangle.*

As seen earlier that in Riemannian case, (0.2.2) implies Soul Conjecture. In comparison, a gap between Theorem 0.5 and (0.4.1) is that for a complete noncompact nonnegatively curved Alexandrov space, there may be points where no flat totally geodesic rectangle obtained in Theorem 0.5 passing through. However in the case that $\text{codim}(S) = 1$ (cf. [37]), Theorem 0.5 implies an affirmative answer to (0.4.2).

In this paper, we will investigate the structure of a complete noncompact nonnegatively curved Alexandrov space which is topologically nice and whose a soul has codimension 2. A point in an Alexandrov space is called topologically nice if the iterated spaces of directions are all homeomorphic to spheres. An Alexandrov space is called topologically nice if all points on it are topologically nice. The limit space of a sequence of noncollapsed Riemannian manifolds with sectional curvature bounded from below uniformly is topologically nice.

We now begin to state the main results in this paper.

Theorem A. *Let A be a complete noncompact nonnegatively curved Alexandrov space, and let $\pi : A \rightarrow S$ be the Sharafutdinov retraction. Suppose that A is topologically nice and that S is of codimension 2. Then π is a submetry.*

We point out that the regularity assumption in Theorem A is used to classify the space of directions of points on S , which is crucial in the proof of Theorem A. Using Theorem A, one easily gets an affirmative answer to Open Problem (0.4.1) in the following 4-dimensional topological manifold case.

Corollary 0.6. *Let A be a complete noncompact nonnegatively curved 4-dimensional Alexandrov space. Suppose A is a topological manifold. If A has positive curvature around a point, then a soul is a point.*

We now explain the main ideas in the proof of Theorem A. We may assume that A is simply connected. (If A is not simply connected, one can pass to the universal cover, see Lemma 1.5.)

For $p \in S$, and $v \in \uparrow_p^{\partial\Omega_c}$ (all directions at p of minimizing geodesics from p to $\partial\Omega_c$), where c is a fixed noncritical value of the Busemann function f (defined in Section 1.1) and $\partial\Omega_c = f^{-1}(c)$, there is always a ray σ at p such that $\sigma^+(0) = v$. We call such ray a special normal ray to S . Let $\mathcal{F} \subseteq A$ be the union of points on all such rays

$$\mathcal{F} = \{x \in A \mid x \in \sigma : \text{a ray with } \sigma(0) = p \in S, \sigma^+(0) = v \in \uparrow_p^{\partial\Omega_c}\}.$$

Observe that in the special case $\mathcal{F} = A$, Theorem A follows easily from Theorem 0.5 (see the proof following Lemma 1.1.).

If $\mathcal{F} \neq A$, we set

$$F_v = \cup \{x \mid x \in \text{flat totally geodesic strips in } A \text{ spanned by } \sigma \text{ and all minimal geodesics in } S \text{ from } p \text{ to all the points in } S\}.$$

We have the following

Key Lemma 0.7. *Let the assumptions be as in Theorem A, assume that A is simply connected. If $\mathcal{F} \neq A$, then for $v \in \uparrow_p^{\partial\Omega_c}$, F_v with the restricted metric isometrically splits, i.e., $F_v \stackrel{\text{isom}}{\cong} S \times \mathbb{R}_+^1$.*

Our proof of Key Lemma 0.7 relies on a property of space of directions on S (Proposition 2.1; note that if we can solve Conjecture 4.5 completely for the

case in Theorem A, then the proof of Theorem A is a little simpler), where the regularity conditions are required. Assuming Key Lemma 0.7, we can choose $F = \cup_{1 \leq i \leq l} F_{v_i}$, $l \leq 3$ such that the distance function from F , dist_F , is concave in $A \setminus F$ (see Lemma 2.8). For any given point $x \in A \setminus F$, let $\hat{x} \in (S, a) \subset F_u \subset F$ such that $|x\hat{x}| = |xF|$. When $a \neq 0$, using the concavity of dist_F , we can construct a “gradient flow” of dist_F from (S, a) passing x , denoted by Ψ_a^t , which is distance nonincreasing (cf. [30]).

Consider the composition $i \circ \pi \circ \Psi_a^t : (S, a) \rightarrow (S, a)$, (where $i : S \rightarrow (S, a)$ is the natural isometry,) which is distance nonincreasing and a deformation, thus onto since $t = 0$ is onto. A standard argument shows that $\pi|_{\Psi_a^{|x\hat{x}|}((S, a))}$ is an isometry. We denote $\Psi_a^{|x\hat{x}|}((S, a))$ by S_x . When $a = 0$, we use a limit argument (see 2.4) to get a $S_x \ni x$, such that $\pi|_{S_x}$ is an isometry.

With the above preparations, we are ready to explain that the Sharafutdinov retraction $\pi : A \rightarrow S$ is a submetry. First π is distance nonincreasing (Theorem 0.3). For any $\bar{y} \in S$, it suffices to find $y \in A$ such that $|xy| = |\pi(x)\bar{y}|$ and $\pi(y) = \bar{y}$. Now it is clear that $y = S_x \cap \pi^{-1}(\bar{y})$ satisfies the desired condition.

Our argument can be viewed as a generalization of [37], where noncompact nonnegatively curved Alexandrov spaces with souls of codimension 1 are classified.

We organize the rest of the paper as follows:

In Section 1, we will collect some basic notions and properties which will be used throughout the paper.

In Section 2, we will prove Theorem A by assuming Proposition 2.1.

In Section 3, some applications are proved there.

In Section 4, we will prove some structural results for spaces of directions of points on S and verify Proposition 2.1 at the end.

1. PRELIMINARIES

We start this section with fixing some notations:

$\text{dist}_x(y) = |xy|$: the distance between points $x, y \in A$

$\text{Alex}^n(\kappa)$: the class of complete n -dimensional Alexandrov spaces with curvature $\geq \kappa$

∂A : the boundary of A , $A \in \text{Alex}^m(\kappa)$

$S^n(\kappa)$: the n -space form of curvature κ

$B(p, r) = \{x \mid |xp| \leq r\}$

$S(p, r) = \{x \mid |xp| = r\}$

$Fr(C)$: the union of points whose any neighborhood contains points in C and in the complement of C

$[pq]$: a minimal geodesic from p to q , $p, q \in A$

\uparrow_p^q : a direction at p of a minimizing geodesic from p to q

\uparrow_p^q : the set of all directions at p of minimizing geodesics from p to q

$\angle(\uparrow_y^x, \uparrow_y^z)$: the angle between \uparrow_y^x and \uparrow_y^z

$\tilde{L}_k(x, y, z)$: the corresponding comparison angle on space form S_2^k

$[CD] = \{x \mid x \in [cd], c \in C, d \in D\}$, where C, D are subsets of A

$C^\perp = \{v \in A \mid |vC| = \frac{\pi}{2}\}$, where $C \subset A$

Flat totally geodesic strip P in X , $X \in \text{Alex}$: P is the image of an isometric embedding from $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, y \geq 0\}$ with the standard flat metric to X

For basic notions related to Alexandrov spaces, we refer to [2], [3], [29], [31] and [35].

In the following, we shall briefly recall the construction of souls using Busemann function and the construction of the Sharafutdinov retractions. Then we shall establish some properties which will be used in our proof or which may not be found in literature.

1.1. Souls and Sharafutdinov retractions.

Throughout this paper, we say a subset C convex, if for any p, q in C , C contains at least one minimal geodesic joining p, q .

Let $A \in \text{Alex}^n(0)$ be noncompact, and let $p \in A$. The Busemann function at p is defined by

$$f(x) = \lim_{t \rightarrow \infty} (|x, S(p, t)| - t),$$

and f is a proper concave function with definite maximum $a_0 = \max_{x \in A} \{f(x)\}$.

Then $C_0 = f^{-1}(a_0)$ satisfies that for any two points, all minimal geodesics joining them are contained in C_0 (and thus C_0 is convex). If $\partial C_0 = \emptyset$, then $C_0 = S$, a soul of A . Otherwise, the distance function, $f_1 = \text{dist}_{\partial C_0} : C_0 \rightarrow \mathbb{R}^1$, is again concave. Let $a_1 = \max_{x \in C_0} \{f_1(x)\}$, and let $C_1 = f_1^{-1}(a_1)$. Repeating the above process for C_1 , and after a finite number of steps we obtain $C_k = S$, a convex subset without boundary.

Next we will recall the construction of a distance nonincreasing deformation retraction from A to S , the so-called Sharafutdinov retraction.

Let $\nabla_q f$ denote the gradient of f at q . Since f is concave, there are f -gradient curves. We reparameterize gradient curves so that a new curve $\alpha(t) \subset (A - C_0)$ satisfies that $\alpha(0) = x$ and $\alpha^+(t) = \frac{\nabla_{\alpha(t)} f}{|\nabla_{\alpha(t)} f|^2}$. Let $\beta(t) \subset (A - C_0)$ be the reparametrization of the gradient curve with $\beta(0) = y$. Without loss of generality, we may assume that $f(x) \leq f(y)$ and $f(\alpha(t_0)) = f(y)$. By a direct computation, we get

$$|\alpha(t)\beta(0)|^+(t) = -\langle \alpha^+(t), \uparrow_{\alpha(t)}^{\beta(0)} \rangle \leq 0, \quad t \leq t_0.$$

Hence

$$|\alpha(t_0)\beta(0)| \leq |\alpha(0)\beta(0)|$$

and

$$|\alpha(t+t_0)\beta(t)|^+(t) = -\langle \alpha^+(t+t_0), \uparrow_{\alpha(t+t_0)}^{\beta(t)} \rangle - \langle \beta^+(t), \uparrow_{\beta(t)}^{\alpha(t+t_0)} \rangle \leq 0.$$

Therefore $|\alpha(t+t_0)\beta(t)| \leq |\alpha(0)\beta(0)|$. From this we can get that $\alpha(t)$ can be uniquely extended to include the points on C_0 , denoted by $\bar{\alpha}(t)$. Define a map,

$\pi_0 : A \rightarrow C_0$, by $\pi_0(x) = \bar{\alpha}(a_0 - f(x))$, with $x = \bar{\alpha}(0)$. We have showed that π_0 is distance nonincreasing.

If $\partial C_0 \neq \emptyset$, repeating the above, we obtain that $\pi_1 : C_0 \rightarrow C_1$ is distance nonincreasing. Eventually, we will get the Sharafutdinov retraction $\pi = \pi_k \circ \dots \circ \pi_0$.

1.2. Flat totally geodesic strips.

The goal of this subsection is to give the following unbounded version of Theorem 0.5, which is known to experts ([42]). Since we can not find a complete proof in literature, for the convenience of readers, we include a proof here.

A useful alternative expression of f is: for any $c < a_0 = \max_{x \in A} \{f(x)\}$, for $x \in \Omega_c = f^{-1}([c, a_0])$, $f(x) = |x\partial\Omega_c| + c$ ([9, Proposition A.1 (5)]).

Lemma 1.1. *Let $A \in \text{Alex}^n(0)$ be noncompact, and let f be a Busemann function. Then the following properties hold:*

(1.1.1) *For $p \in S$, let $q \in \partial\Omega_{f(q)}$ such that $|pq| = |p\partial\Omega_{f(q)}|$. Then $[pq]$ can be extended to a ray γ , with $\gamma(0) = p$ and $|p\gamma(t)| = |p\partial\Omega_{f(\gamma(t))}|$, for any $t \geq 0$.*

(1.1.2) *For $p \neq r \in S$, there exists a ray σ with $\sigma(0) = r$ and $|r\sigma(t)| = |r\partial\Omega_{f(\sigma(t))}|$, for any $t \geq 0$, and $\{\gamma, [pr], \sigma\}$ bounds a flat totally geodesic strip.*

Proof of Theorem A for the case that $\mathcal{F} = A$. For any $x \in A$, we have that $x \in \gamma$: a special normal ray from \bar{x} . Hence $\pi(x) = \bar{x}$ (see Lemma (1.1.1)). For any $\bar{y} \in S$, by Lemma (1.1.2), there is a flat totally geodesic strip determined by $\{\gamma, [\bar{x}\bar{y}]\}$; in which we can find $y \in \pi^{-1}(\bar{y})$, such that $|\bar{x}\bar{y}| = |xy|$. \square

In the proof of Lemma 1.1 we will use the following lemma.

Lemma 1.2 ([42, 2.5]). *Let $\Sigma \in \text{Alex}^n(1)$, and let $C \subset \Sigma$ be a locally convex closed subset without boundary with positive dimension. If $v \in \Sigma$ such that $|vC| \geq \frac{\pi}{2}$, then $|v\xi| = \frac{\pi}{2}$, for any $\xi \in C$.*

We emphasize that Lemma 1.2 will be frequently used throughout the paper.

Let X be an Alexandrov space. For $p \in X$, let $T_p X$ (or T_p) denote the tangent cone of X at p , and let $\Sigma_p X$ (or Σ_p) denote the space of directions of X at p .

Proof of Lemma 1.1. (1.1.1): Let $q_1 \in A$ such that $f(q_1) < f(q)$ and $|qq_1| = |q\partial\Omega_{f(q_1)}|$. Then $f(p) - f(q_1) = |p\partial\Omega_{f(q)}| + |q\partial\Omega_{f(q_1)}| \geq |pq| + |qq_1| \geq |pq_1| \geq |p\partial\Omega_{f(q_1)}| = f(p) - f(q_1)$. Thus $[pq] \cup [qq_1]$ is a minimal geodesic with the desired property. Iterating this process, one can get the desired ray $\gamma(t)$.

(1.1.2): Note that $|pq| = |p\partial\Omega_{f(q)}|$ implies that $|pq| = |qS|$. Then by the first variation formula, $|\uparrow_p^q \Sigma_p S| \geq \frac{\pi}{2}$, and by Lemma 1.2, $|\uparrow_p^q v| = \frac{\pi}{2}$, for any $v \in \Sigma_p S$. Thus by Theorem 0.5, for $t_1 > 0$, there exists a flat totally geodesic rectangle P_1 with two of the edges $[p\gamma(t_1)]$ and $[pr]$. Hence there is a corresponding point $r_1 \in \partial\Omega_{f \circ \gamma(t_1)}$ such that $|rr_1| = |r\partial\Omega_{f \circ \gamma(t_1)}|$. By applying Theorem 0.5, we get another flat totally geodesic rectangle P_2 with two of the edges $[\gamma(t_1)\gamma(t_2)]$ and $[\gamma(t_1)r_1]$, there is a corresponding point $r_2 \in \partial\Omega_{f \circ \gamma(t_2)}$, such that $|r_1r_2| = |r_1\partial\Omega_{f \circ \gamma(t_2)}|$. Next we will show that $P_1 \cup P_2$ is a flat totally geodesic rectangle. There is a

canonical map, $g : R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq |pr|, 0 \leq y \leq t_2\} \rightarrow P_1 \cup P_2$, with $g((0, 0)) = p$, and $g(\{x = 0, 0 \leq y \leq t_2\}) = \gamma$. In order to show that g is an isometry, it suffices to show that $|g(z_1)g(z_2)| = |z_1z_2|$, for any $z_1, z_2 \in \partial R$. First one can easily see that $g(\{x, 0 \leq y \leq t_2\})$ are all minimal geodesics, i.e., vertical direction are all isometry. The left cases are similar. We just show the case of $z_1 = (0, t_2)$ and $z_2 = (|pr|, 0)$. For $\gamma(t), 0 \leq t \leq t_2$ and $[pr]$, we can apply Theorem 0.5 to get another flat totally geodesic rectangle, thus we get that $|\gamma(t_2)r| = \sqrt{t_2^2 + |pr|^2}$, then $|g(z_1)g(z_2)| = |z_1z_2|$ follows.

Let $P = \cup_{i \geq 0} P_i$. It follows that P is a flat totally geodesic strip and there is a corresponding geodesic ray from r with the desired property. \square

Note that flat totally geodesic strip in Lemma (1.1.2) may not be unique, see example [42, 14.8].

Remark 1.3. Inspecting the proof of Lemma 1.2, one can see that when $\partial C \neq \emptyset$ the following holds: Let $x \in C$ be a point such that $|vC| = |vx|$. If $x \notin \partial C$, then $|v\xi| = \frac{\pi}{2}$, for any $\xi \in C$. We will use this observation in Section 4.

1.3. A reduction.

The goal here is to reduce the proof of Theorem A to the simply connected case.

Let $A \in \text{Alex}^n(\kappa)$. Recall that $p \in A$ is topologically regular, if $\Sigma_p A$ is homeomorphic to a sphere. A topologically regular point has a neighborhood homeomorphic to a Euclidean ball. A is called topologically regular, if all points are topologically regular, and thus A is a topological manifold.

A point $p \in A$ is called topologically nice, if the iterated spaces of directions, i.e., $\Sigma_p A, \Sigma_{v_1} \Sigma_p A, \dots$ are all homeomorphic to spheres. A is called topologically nice, if all points are topologically nice. Topologically nice implies topologically regular, but the converse may not be true.

Example 1.4 ([21]). Let Σ^3 be the Poincaré homology 3-sphere with constant curvature 1. Then the three-fold spherical suspension of Σ^3 , $S^3(\Sigma^3) \in \text{Alex}^6(1)$, is topologically regular but not topologically nice.

Lemma 1.5. *If Theorem A holds for simply connected Alexandrov spaces, then it holds for any non-simply connected Alexandrov spaces.*

Proof. Let \tilde{S} be the universal cover of S . Denote the covering map by ϕ . Let $\phi^*(A) = \{(p, e) \in \tilde{S} \times A \mid \phi(p) = \pi(e)\} \subset \tilde{S} \times A$, with the induced topology. Then by a standard argument, we obtain that $\phi^*(A)$ is the universal cover of A , and $\tilde{\phi} : \phi^*(A) \rightarrow A$, defined by $\tilde{\phi}((p, e)) = e$, is the covering map.

Endow $\phi^*(A)$ with the induced metric, denoted by \tilde{A} . Then $\tilde{\phi} : \tilde{A} \rightarrow A$ is a local isometry, and $\tilde{\pi} : \tilde{A} \rightarrow \tilde{S}$, with $\tilde{\pi}((p, e)) = p$, is locally 1-Lipschitz.

First we assume that $C_0 = S$.

Sublemma 1.6. *Let $\tilde{\Omega}_c = \{(p, e) \in \tilde{S} \times \Omega_c \mid \phi(p) = \pi(e)\}$. Then $\tilde{\Omega}_c \subset \tilde{A}$ is convex.*

Proof. For $x, y \in \tilde{\Omega}_c$, if $[xy] \subsetneq \tilde{\Omega}_c$, by the construction of \tilde{A} , there exists a curve in $\tilde{\Omega}_c$ with length $\leq |xy|$, a contradiction. Thus we get the sublemma. \square

Let $\tilde{f} = \text{dist}_{\partial\tilde{\Omega}_c}$. By the property of covering space, $\tilde{f}((p, e)) = |(p, e), \partial\tilde{\Omega}_c| = |e, \partial\Omega_c| = f(e) - c$. It follows that $\partial\tilde{\Omega}_c$ are level sets of \tilde{f} . And by the local isometry, $|\nabla \tilde{f}_{(p,e)}| = |\nabla f_e|$. Hence $\alpha(t)$ is an f -gradient curve if and only if $(p, \alpha(t))$ is a \tilde{f} -gradient curve.

If \tilde{S} is compact, by the assumption of the lemma, we can see that $\tilde{\pi}$ is a submetry (since \tilde{A} is topologically nice). Hence π is also a submetry.

If \tilde{S} is not compact, by the splitting theorem [23], there is an isometric splitting $\tilde{S} = \mathbb{R}^k \times S_0$, where S_0 is simply connected and compact, exactly as the proof of Riemannian case. It follows that $\tilde{A} = \mathbb{R}^k \times A_0$ and $\partial\tilde{\Omega}_c = \mathbb{R}^k \times \partial\Omega'_c$. We claim that $\nabla_{(x,x_0)} \text{dist}_{\partial\tilde{\Omega}_c} \in T_{x_0}A_0$, for any $(x, x_0) \in \tilde{A}$. Hence $\tilde{\pi} = (id, \pi_0)$.

Since $\tilde{A} = \mathbb{R}^k \times A_0$ is topologically nice, we have that A_0 is topologically nice (see Remark 1.7), as can be seen in the proof of [32, Theorem D]. Thus by the assumption of the lemma, we know that $\pi_0 : A_0 \rightarrow S_0$ is a submetry. It follows that $\tilde{\pi}$ is a submetry.

Finally, we will verify the claim: for any $x_0 \in A_0$ and $y = (y_1, x_0) \in \mathbb{R}^k \times A_0$, we have that $\Sigma_y \tilde{A} = S^k(\Sigma)$, where $\Sigma = \Sigma_{x_0}A_0$. Let v be a point such that $|v \uparrow_y^{\partial\tilde{\Omega}_c}| = \max_{w \in \Sigma_y \tilde{A}} \{| \uparrow_y^{\partial\tilde{\Omega}_c} w |\}$. Since $\partial\tilde{\Omega}_{\bar{c}} = \mathbb{R}^k \times \partial\Omega'_{\bar{c}}$, for any $\bar{c} < a_0$, we have that $\uparrow_y^{\partial\tilde{\Omega}_c} \in \Sigma$. It follows that $v \subset \Sigma$. By the definition of gradient the claim follows.

If $C_0 \neq S$, consider $\text{dist}_{\partial\Omega_{a_0}}$ instead of f , we can get the same conclusion. \square

Remark 1.7. If A is only topologically regular, then A_0 may not be a topological manifold, even $\tilde{A} = \mathbb{R}^k \times A_0$ is a topological manifold ([21]).

2. PROOF OF THEOREM A

In our proof of Theorem A, the following structural results on spaces of directions of points on soul plays a curial role.

Proposition 2.1. *Let the assumptions be as in Theorem A. For $p \in S$, let $\Sigma_0^p = \Sigma_p S$, and let $\Sigma_1^p = \{v \in \Sigma_p A \mid |v \Sigma_p S| \geq \frac{\pi}{2}\}$. Then*

(2.1.1) Σ_0^p is homeomorphic to a sphere.

(2.1.2) Σ_1^p is convex¹ and isometric to one of the following:

$S^1(r)$ with $r \leq 1$, $[ab]$, $\{v\}$, $\{v_1, v_2\}$ with $|v_1 v_2| = \pi$.

(2.1.3) With the restricted metric, $[\Sigma_0^p \Sigma_1^p] = \Sigma_0^p * \Sigma_1^p$ with the standard join metric².

(2.1.4) When $\Sigma_1^p = [ab]$, if there exists a subset E of Σ_1^p such that $B(E, \frac{\pi}{2}) = \Sigma_p A$, then $a, b \in E$. If $|ab| = \pi$ with o the middle point of $[ab]$, then for any $x \in \Sigma_p A$ such that $|ox| \leq \frac{\pi}{2}$, $x \in [\Sigma_0^p \Sigma_1^p]$.

¹See Remark 4.3 (2), (1).

²See Definition 4.1

Because the proof of Proposition 2.1 is technical and long, we will postpone the proof to the next section. Below we shall prove Theorem A by assuming Proposition 2.1.

2.1. Proof of Key Lemma 0.7.

Recall that

$F_v = \cup \{x \mid x \in \text{flat totally geodesic strips in } A \text{ spanned by } \sigma \text{ and all minimal geodesics in } S \text{ from } p \text{ to all the points in } S\}.$

where $p \in S$, $v \in \uparrow_p^{\partial\Omega_c} \subset \Sigma_1^p$ ($c < \max f$), $\sigma(t) \subset A$ is a ray with $\sigma(0) = p$ and $\sigma^+(0) = v$ (see Lemma (1.1.1)), and if $\gamma \subset S$ is a minimal geodesic from p to $q \in S$, then $\{\sigma, \gamma\}$ determines a unique flat totally geodesic strip (see Lemma (1.1.2)); the uniqueness follows from Proposition (2.1.3), otherwise, will violate the join of $[\Sigma_0^p \Sigma_1^p]$.

Let α be the other ray from q which bounds the flat totally geodesic strip, with $\bar{v} = \alpha^+(0) \in \uparrow_q^{\partial\Omega_c}$. We then define a map $\phi_{[pq]} : \uparrow_p^{\partial\Omega_c} \rightarrow \uparrow_q^{\partial\Omega_c}$, by $\phi_{[pq]}(v) = \bar{v}$. Note that $\phi_{[pq]}$ may depend on the choice of $[pq]$.

Observe that for all $q \in S$, $\phi_{[pq]}$ is independent of $[pq]$ if and only if F_v is isometric to $S \times \mathbb{R}_+^1$, i.e., Key Lemma 0.7 holds.

To prove the independency, we will first show that $\phi_{[pq]}$ is an isometry. (We point out that the method we used in the proof of Lemma 2.2 below was previously used in [29].)

Lemma 2.2. *Let the assumptions be as in Theorem A. For every $x \in S$ and every $y \in S$, $\phi_{[xy]} : \uparrow_x^{\partial\Omega_c} \rightarrow \uparrow_y^{\partial\Omega_c}$, is an isometry, for any minimal geodesic $[xy]$.*

Proof. For $u, v \in \uparrow_x^{\partial\Omega_c}$, and $\varepsilon > 0$, let $\bar{v}_\varepsilon = |y, \exp_x(\varepsilon v)| \uparrow_y^{\exp_x(\varepsilon v)}$, where $\exp_x : T_x \rightarrow A$ is the usual exponential map, and $\uparrow_y^{\exp_x(\varepsilon v)}$ is the direction of the minimal geodesic contained in the flat totally geodesic strip determined by $\exp_x(tv), t \geq 0$ and $[xy]$. Similarly for \bar{u}_ε , with εu instead of εv in the definition of \bar{v}_ε .

Since by Proposition (2.1.3) and by the property of flat totally geodesic strip, we have that $|\bar{v}_\varepsilon^\perp \bar{u}_\varepsilon^\perp|_{T_y} = |\bar{v}_\varepsilon \bar{u}_\varepsilon|_{T_y} \geq |\exp_x(\varepsilon v) \exp_x(\varepsilon u)| = \varepsilon |uv|_{T_x} + o(\varepsilon)$, where \perp is the orthogonal projection to $C(\Sigma_1^y)$ (Euclidean cone over Σ_1^y).

Let $\bar{u} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \bar{u}_\varepsilon^\perp = \phi_{[xy]}(u)$ and $\bar{v} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \bar{v}_\varepsilon^\perp = \phi_{[xy]}(v)$. Then $|\bar{u}\bar{v}|_{T_y} \geq |uv|_{T_x}$, and $|\bar{u}| = |u|, |\bar{v}| = |v|$. Thus we get that $|\phi_{[xy]}(u)\phi_{[xy]}(v)|_{\Sigma_y} \geq |uv|_{\Sigma_x}$.

Similarly, the opposite inequality holds. Hence $|\phi_{[xy]}(u)\phi_{[xy]}(v)|_{\Sigma_y} = |uv|_{\Sigma_x}$. \square

Remark 2.3. It seems that Lemma 2.2 can be strengthened to that the isometric class of Σ_1^p is independent of p .

Proof of Key Lemma 0.7. Define a map, $\psi : F_v \rightarrow S \times \mathbb{R}_+^1$, $\psi(\exp_q(t\phi_{[pq]}(v))) = (q, t)$. As pointed out earlier, ψ is an isometry if $\phi_{[pq]}(v)$ is independent of $q \in S$. In view of the simply connectedness of S (because A is simply connected), first we will show that F_v is a product locally, it suffices to show that locally $\phi_{[pq]}(v)$ is independent of $q \in S$ (all p). Precisely, for $x \in S$, there exists $\varepsilon > 0$ (ε depends on x), such that for any $y, z \in B(x, \varepsilon)$, $g = \phi_{[zx]} \circ \phi_{[yz]} \circ \phi_{[xy]} = id$.

If $\Sigma_1^x = \{v\}$, then $\uparrow_x^{\partial\Omega_c} = \Sigma_1^x$. Thus each point in S has just one special normal ray to S , clearly $g = id$.

For other cases we will argue by contradiction. Suppose that for a sequence $\varepsilon_i \rightarrow 0$, there exist $y_i, z_i \in B(x, \varepsilon_i)$, $g_i = \phi_{[z_i x]} \circ \phi_{[y_i z_i]} \circ \phi_{[x y_i]} \neq id$.

If $\Sigma_1^x = \{v_1, v_2\}$ with $|v_1 v_2| = \pi$, then $g_i(v_1) = v_2$, and by the property of flat totally geodesic strips, we have that $|\exp_x(v_1) \exp_x(g_i(v_1))| \leq (|xy| + |yz| + |zx|) \leq 4\varepsilon_i$. When $\varepsilon_i \rightarrow 0$, we get a contradiction, since $|\exp_x(v_1) \exp_x(v_2)| > 0$, or geodesic branches.

If $\Sigma_1^x = S^1(r)$ with $r \leq 1$, by Lemma 2.4 below, we have that every g_i is the restriction of an isometry, $\bar{g}_i : S^1 \rightarrow S^1$, which is a rotation or a reflection. By passing to a subsequence, we can suppose that every \bar{g}_i is a rotation or every \bar{g}_i is a reflection.

(a): Every \bar{g}_i is a rotation.

For $v \in \uparrow_x^{\partial\Omega_c}$, $|\exp_x(tv) \exp_x(g_i(tv))| \rightarrow 0$, for any $t \geq 0$, as $\varepsilon_i \rightarrow 0$, which can be seen in the above case. Hence $|vg_i(v)| \rightarrow 0$, i.e. $|v\bar{g}_i(v)| \rightarrow 0$, as $\varepsilon_i \rightarrow 0$. Then by the closeness of $\uparrow_x^{\partial\Omega_c}$, we can get that $\uparrow_x^{\partial\Omega_c} = S^1$. This is a contradiction, since $\mathcal{F} \neq A$ and Lemma 2.2 imply that $\uparrow_q^{\partial\Omega_c} \neq S^1$, for any $q \in S$.

(b): Every \bar{g}_i is a reflection.

By passing to a subsequence, we can assume that $\bar{g}_i \rightarrow h$, which is also a reflection. Observe that there is $v \in \uparrow_x^{\partial\Omega_c}$, such that $v \neq h(v)$, or g_i will be equal to id , a contradiction. Similarly, we have that $|\exp_x(tv) \exp_x(g_i(tv))| \rightarrow 0$, for any $t \geq 0$, as $\varepsilon_i \rightarrow 0$. Hence $|vg_i(v)| \rightarrow 0$, i.e. $|v\bar{g}_i(v)| \rightarrow 0$, as $\varepsilon_i \rightarrow 0$. Thus we have that $v = h(v)$, a contradiction.

If $\Sigma_1^x = [ab]$, likewise by Lemma 2.4, each g_i just can be the restriction of the reflection of $[ab]$. Similarly as above, we can get the conclusion.

Then $F_v|_{B(x, \varepsilon)}$ is a product. (We call $B(x, \varepsilon)$ a local product neighborhood of x .)

Finally, we will show that F_v is a product globally. For any $q, r \in S$, and for three fixed geodesics $[xq], [qr]$ and $[rx]$, let $\gamma = [xq] \cup [qr] \cup [rx]$. Since $\pi_1(S) = 0$, γ is homotopic to a point. Let $H : [0, 1] \times [0, 1] \rightarrow S$ be a homotopy, with $H(t, 1) = \gamma(t)$, $H(t, 0) = x$, $H(0, s) = x$ and $H(1, s) = x$. Let $\{s_0 = 0 < s_1 < \dots < s_n = 1\}$ and $\{t_0 = 0 < t_1 < \dots < t_n = 1\}$ be two partitions of $[0, 1]$, such that $H([s_i, s_{i+1}] \times [t_j, t_{j+1}]) \subset U_z$; a closed convex neighborhood of some $z \in S$ which is contained in the local product neighborhood of z . Let $\sigma_i = \cup_{j=0}^{n-1} [H(s_i, t_j) H(s_i, t_{j+1})]$. There are corresponding g_i . We can see that $g_i(w)$, for any $w \in \uparrow_x^{\partial\Omega_c}$, are the same for any i . It follows that $g_1(w) = w$, i.e., $g_1 = id$. Hence $\phi_{[rx]} \circ \phi_{[qr]} \circ \phi_{[xq]} = id$. Thus we can get that ψ is an isometric map from $S \times \mathbb{R}_+^1$ to F_v . \square

Lemma 2.4. Let $M \stackrel{\text{homeo}}{\cong} S^1$ or an interval and with intrinsic metric, let $N \subset M$ be a subset and let $g : N \rightarrow N$ be an isometry, where N is with the restricted metric. Then g can be extended to an isometry $\bar{g} : M \rightarrow M$.

Proof. We just show the case of $M \stackrel{\text{homeo}}{\cong} S^1$, similarly for an interval.

If there exist $v, w \in N$ such that v, w are not antipodal, then for any $u \in M$, u is uniquely determined by $|uv|, |uw|$. Thus g is uniquely determined by $g(v), g(w)$. Hence g can be extended to \bar{g} , by $\bar{g}(u) = x$, where x is the unique point such that $|xg(v)| = |uv|$ and $|xg(w)| = |uw|$.

If not, then $N = \{v, w\}$ with v, w antipodal. Clearly g is extendable. \square

Next we will show

Lemma 2.5. *For every $x \in F_u$ and every $y \in F_u$, $[xy] \subset F_u$.*

In the proof of Lemma 2.5, we need the following lemma.

Lemma 2.6 ([14, 2.4(ii')]). *Let $X \in \text{Alex}^m(\kappa)$. For two minimal geodesics $[xz]$ and $[xy]$, if $\angle(\uparrow_x^z, \uparrow_x^y) = \tilde{\angle}(z, x, y)$, then there is a $[zy]$ such that $[xz], [xy], [zy]$ bound a totally geodesic surface which is isometric to a geodesic triangle in $S^2(\kappa)$.*

Proof of Lemma 2.5. Set $\pi(x) = \bar{x}$ and $\pi(y) = \bar{y}$. If $x, y \in S$, by the construction of S , we have that $[xy] \subset S$.

For other cases we will argue by contradiction. Suppose that there exist $x, y \in F$ such that there exists $[xy]$ which doesn't belong to F . Then $[xy]^\circ \cap F = \emptyset$, where $[xy]^\circ$ denotes $[xy] - \{x, y\}$, or geodesic will branch.

If $x \in S, y \in (S, a)$, with $a \neq 0$, let $r \in F_u, r \neq y$ be a point such that $\pi(r) = \bar{y}$ and $|r\pi(r)| > |y\bar{y}|$. Then $\pi = \angle(\uparrow_y^x, \uparrow_y^{\bar{y}}) + \angle(\uparrow_y^x, \uparrow_y^r) \geq \tilde{\angle}(x, y, \bar{y}) + \tilde{\angle}(x, y, r) = \pi$, where the last equality is from the construction of F . Hence $\angle(\uparrow_y^x, \uparrow_y^{\bar{y}}) = \tilde{\angle}(x, y, \bar{y})$. It follows from Lemma 2.6 that $\{x, y, \bar{y}\}$ bounds another flat totally geodesic triangle, which contradicts to the structure of $\Sigma_{\bar{y}}A$ (Proposition (2.1.3)), since $[x\bar{y}] \subset S$.

If $x, y \in (S, a)$, with $a \neq 0$, then the same as above we have that $\angle(\uparrow_x^y, \uparrow_x^{\bar{x}}) = \tilde{\angle}(y, x, \bar{x}) = \frac{\pi}{2}$. Therefore there exists a flat triangle bounded by y, x, \bar{x} for the given $[xy]$, with $[y\bar{x}]^\circ \not\subset F$. By the above case, we get a contradiction.

If $x \in (S, a), y \in (S, b)$, with $a \neq 0, b \neq 0$ and $a \neq b$, without loss of generality, we can assume that $a < b$. Let $s = [y\bar{y}] \cap (S, a)$. Similarly as the above two cases, we can also get a contradiction. \square

As seen following Lemma 1.2, the remaining case in the proof of Theorem A is that $\mathcal{F} \neq A$ and $\pi_1(A) = 0$, which implies that $\uparrow_p^{\partial\Omega_c} \neq S^1$.

2.2. The concavity of dist_F .

As seen in the introduction, F is the union of several F_v 's. We point it out that the selection of these F_v 's is crucial for the desired concavity of dist_F ; see following for details.

For $p \in S$, by the first variation formula for the Busemann function, $d_p f(v) = -\langle \uparrow_p^{\partial\Omega_c}, v \rangle \leq 0$, $v \in \Sigma_p$, we see that $\uparrow_p^{\partial\Omega_c}$ is $\frac{\pi}{2}$ -dense in Σ_p and thus $\frac{\pi}{2}$ -dense in Σ_1^p .

Lemma 2.7. *There is $N' = \{v_i\}_{1 \leq i \leq l} \subset \uparrow_p^{\partial\Omega_c}, l \leq 3$, such that N' is $\frac{\pi}{2}$ -dense in Σ_1^p , and $\phi_{[pq]}(N')$ is also $\frac{\pi}{2}$ -dense in Σ_1^q , for any $q \in S$.*

Proof. First for the selection of N' : if $\Sigma_1^p = \{v\}$, then $\uparrow_p^{\partial\Omega_c} = \Sigma_1^p$. Let $N' = \uparrow_p^{\partial\Omega_c}$.

If $\Sigma_1^p = \{v_1, v_2\}$ with $|v_1 v_2| = \pi$, then $v_1, v_2 \in \uparrow_p^{\partial\Omega_c}$. Indeed, since $\uparrow_p^{\partial\Omega_c} \neq \emptyset$, one of them, say v_1 , must be in $\uparrow_p^{\partial\Omega_c}$. Suppose that $v_2 \notin \uparrow_p^{\partial\Omega_c}$, then $d_p f(v_2) = 1 > 0$, a contradiction. Let $N' = \uparrow_p^{\partial\Omega_c}$.

If $\Sigma_1^p = [ab]$, since $B(\uparrow_p^{\partial\Omega_c}, \frac{\pi}{2}) = \Sigma_p A$, by Proposition (2.1.4), we have that $a, b \in \uparrow_p^{\partial\Omega_c}$. Let $N' = \{a, b\}$. Then N' is obviously a $\frac{\pi}{2}$ -dense subset of Σ_1^p .

If $\Sigma_1^p = S^1(r)$ with $r \leq 1$, we choose $v \in \uparrow_p^{\partial\Omega_c}$ arbitrarily, consider the antipodal point of v , denoted by w . If $w \in \uparrow_p^{\partial\Omega_c}$, let $N' = \{v, w\}$. If $w \notin \uparrow_p^{\partial\Omega_c}$, let $v_1, v_2 \in \uparrow_p^{\partial\Omega_c}$ be the farthest points to w from both sides respectively in Σ_1^p , it follows that v, v_1, v_2 (v may be equal to v_1 or v_2) form a $\frac{\pi}{2}$ -dense subset of Σ_1^p , let $N' = \{v, v_1, v_2\}$.

For the second part of the lemma: if $\Sigma_1^p = \{v\}$ or $\{v_1, v_2\}$, by Lemma 2.2 one can deduce that N' has the desired property.

If $\Sigma_1^p = [ab]$, suppose that there is $q \in S$ such that $\phi_{[pq]}(N')$ is not a $\frac{\pi}{2}$ -dense subset of Σ_1^q . then there is $w \in \uparrow_p^{\partial\Omega_c}$ such that $|\phi_{[pq]}(w)\phi_{[pq]}(a)| + |\phi_{[pq]}(w)\phi_{[pq]}(b)| > |ab|$, a contradiction, since by Lemma 2.2, $\phi_{[pq]}$ is an isometry when restricted to $\uparrow_p^{\partial\Omega_c}$.

If $\Sigma_1^p = S^1(r)$ with $r \leq 1$, then from the choosing method, we have that for any $u \in \uparrow_p^{\partial\Omega_c}$, either $|uv| + |uv_1| = |vv_1|$, or $|uv| + |uv_2| = |vv_2|$. Suppose that there is $q \in S$ such that $\phi_{[pq]}(N')$ is not a $\frac{\pi}{2}$ -dense subset of Σ_1^q . Then there is $w \in \uparrow_p^{\partial\Omega_c}$ such that $|\phi_{[pq]}(w)\phi_{[pq]}(v)| + |\phi_{[pq]}(w)\phi_{[pq]}(v_1)| > |vv_1|$, and $|\phi_{[pq]}(w)\phi_{[pq]}(v)| + |\phi_{[pq]}(w)\phi_{[pq]}(v_2)| > |vv_2|$, a contradiction.

Thus we finish the proof of the lemma. \square

Lemma 2.8. *Let $F = \cup_{v \in N'} F_v$. The distance function, dist_F , is concave in $D = X \setminus F$.*

Remark 2.9. Observe that if the boundary points of each component D_i are “true” boundary points, i.e., which are not interior points in the closure \bar{D}_i and that \bar{D}_i is convex, then it follows that dist_F is concave in D . In our case, we show that even if a component of D may not be convex, dist_F is still concave. For example: let $T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 10x\} \cup \{(x, y) \mid x \leq 0, y \geq x^2\}$, the metric product, $A = \text{Doub}(T) \times S^2(1) \in \text{Alex}^4(0)$. Then $F = \text{Doub}(\{x = 0, y \geq 0\})$ serves an example.

In the proof of Lemma 2.8, we need the following lemma which is an analogue to the totally geodesic property in Riemannian geometry. (Lemma 2.10 below is from a helpful discussion with Shicheng Xu.)

Lemma 2.10. *Let $X \in \text{Alex}^n(\kappa)$, and let Y be a closed subset of X such that for $x, y \in Y$, any $[xy] \subset Y$. Then for any $p \in Y$ and $q \in Y \setminus \partial Y$, we have that $\nabla_q \text{dist}_p \in T_q Y$.*

We don't know whether Lemma 2.10 is true for convex subset or not.

Proof of Lemma 2.10. If $\nabla_q \text{dist}_p = 0$, nothing need to prove. Hence we can assume $\nabla_q \text{dist}_p \neq 0$. Since $\frac{\nabla_q \text{dist}_p}{|\nabla_q \text{dist}_p|} = \{u \in \Sigma_q X \mid |u \uparrow_q^p| = \max_{v \in \Sigma_q} \{|v \uparrow_q^p|\}\}$, and by

the condition of the lemma, we have that $\uparrow_q^p \subset \Sigma_q Y$, it suffices to show that for $V \subset \Sigma_q Y$, let $w \in \Sigma_q X$ be a point such that $|Vw| = \max\{|V, |\} > \frac{\pi}{2}$, then we have that $w \in \Sigma_q Y$.

Argue by contradiction. Suppose that $w \notin \Sigma_q Y$. Choose $w_0 \in \Sigma_q Y$, such that $|ww_0| = |w\Sigma_q Y|$. By Lemma 1.2, $|\uparrow_{w_0}^w \bar{v}| = \frac{\pi}{2}$, for any $\bar{v} \in \Sigma_{w_0} \Sigma_q Y$ and $|ww_0| \leq \frac{\pi}{2}$. Hence $\tilde{Z}(\tilde{w}, \tilde{w}_0, \tilde{V}) \leq \frac{\pi}{2}$. By hinge comparison, we have that $|Vw_0| > \frac{\pi}{2}$. It follows that $|Vw| < |Vw_0|$, a contradiction to the choice of w . \square

Recall a standard fact in topology (cf. [5]): If $X \subset S^m$ is a closed $(m-1)$ -topological manifold as a subspace, then $S^m - X$ has two connected components, each having X as its set boundary. We say that X separates S^m .

Proof of Lemma 2.8. Given $q \in D$, let $\gamma(t) \subset D$ be a minimal geodesic with $\gamma(0) = q$, and let $x \in F$ be a point such that $|qx| = |qF|$. By a standard contradiction argument, one can get that for $v = \uparrow_x^{\gamma(t)} \in \Sigma_x A$, when t is small enough, there is \uparrow_x^q such that $|v \uparrow_x^q| < \frac{\pi}{2}$. (Note that for different t , \uparrow_x^q may be different.)

From the proof of the concavity of distance function to the boundary of an Alexandrov space (cf. [30, Theorem 3.3.1], [9, Lemma 3.1]), one can deduce that if F satisfies the following two conditions:

- (i) there is $\bar{w} \in \Sigma_x F$ such that $|\uparrow_x^q \bar{w}| = |\uparrow_x^q v| + |v\bar{w}| = \frac{\pi}{2}$,
- (ii) there is a radial curve, $\sigma : [0, \varepsilon] \rightarrow F$, with $\sigma(0) = x$ and $\sigma^+(0) = \bar{w}$, for some $\varepsilon > 0$,

then dist_F is concave in D .

Thus it suffices to check that F satisfies the two conditions.

For condition (i): if $x \in S$, by the first variation formula, we have that $|\uparrow_x^q \Sigma_x F| \geq \frac{\pi}{2}$. Specially, $|\uparrow_x^q \Sigma_x S| \geq \frac{\pi}{2}$, thus $\uparrow_x^q \in \Sigma_1^x$. Hence $\Sigma_1^x = S^1$ or $[ab]$ with $|ab| = \pi$ and \uparrow_x^q the middle point of $[ab]$. And by Lemma 2.7, we have that for any $v \in \Sigma_x$, $|v\phi_{[px]}(N')| \leq \frac{\pi}{2}$. Hence there are two of N' say v_1, v_2 (v_1 may be equal to v_2) such that $|\phi_{[px]}(v_1) \uparrow_x^q| = \frac{\pi}{2}$, $|\phi_{[px]}(v_2) \uparrow_x^q| = \frac{\pi}{2}$. Since $|\uparrow_x^q v| < \frac{\pi}{2}$, by Proposition (2.1.4), we have that $v \in [\Sigma_0^x \Sigma_1^x]$. Thus by Proposition (2.1.3), we have that $v \in [w_1 w_0]$, for some $w_i \in \Sigma_i^x$. We can suppose that $w_1 \in [\uparrow_x^q \phi_{[px]}(v_2)]$. Then $\Delta(\uparrow_x^q, w_0, \phi_{[px]}(v_2))$ is isometric to a triangle with three side lengths $\frac{\pi}{2}$ on $S^2(1)$. Hence there is $\bar{w} \in \Sigma_x F$ such that $|\uparrow_x^q \bar{w}| = |\uparrow_x^q v| + |v\bar{w}| = \frac{\pi}{2}$.

If $x \in S$, $\Sigma_x F = S(\Sigma_x S) \stackrel{\text{homeo}}{\simeq} S^{n-2}$ is convex in $\Sigma_x A$ and separates $\Sigma_x A$, and by the first variation formula, $|\uparrow_x^q \Sigma_x F| \geq \frac{\pi}{2}$. Then by Lemma 4.19 below, we have that there is $\bar{w} \in \Sigma_x F$ such that $|\uparrow_x^q \bar{w}| = |\uparrow_x^q v| + |v\bar{w}| = \frac{\pi}{2}$.

For condition (ii): since $\bar{w} \in \Sigma_x F$, without loss of generality, we can assume that $\bar{w} \in \Sigma_x F_{v_1}$. Since $(\Sigma_x F_{v_1})'$ is dense in $\Sigma_x F_{v_1}$, there are $q_i \in F_{v_1}$, such that $q_i \rightarrow x$ and $\uparrow_x^{q_i} \rightarrow \bar{w}$. Let σ_i be the radial curve at q_i with respect to x . By [1, Chapter 15] or [27, 3.4], we know that if we can show that $\lim_{i \rightarrow \infty} \sigma_i([0, \varepsilon]) = \sigma([0, \varepsilon]) \subset F_{v_1}$ for some small $\varepsilon > 0$, then σ is the desired radial curve.

If $x \in S$, by Lemma 2.10, we can get the desired radial curve. If $x \in S$ and $\bar{w} \in \Sigma_x S$, similarly by Lemma 2.10, we can get the desired radial curve in S . If

$x \in S$ and $\bar{w} \notin \Sigma_x S$, we can choose $q_i \in F_{v_1} - S$. We claim that $\sigma_i([0, \infty)) \subset F_{v_1}$. Therefore we can get the desired radial curve.

Finally, we will verify the claim by showing that σ_i are more and more farther away from ∂F_{v_1} . The reason is that $(\text{dist}_S \circ \sigma_i(t))^+ = -\langle \uparrow_{\sigma_i(t)}^{\pi(\sigma_i(t))}, \sigma_i^+(t) \rangle = -\langle \uparrow_{\sigma_i(t)}^{\pi(\sigma_i(t))}, \frac{|x\sigma_i(t)|}{t} \nabla_{\sigma_i(t)} \text{dist}_x \rangle \geq 0$, the last inequality is because of the symmetry of F locally, we have that \uparrow_q^x are in the same half sphere as $\uparrow_q^{\pi(q)} = \uparrow_q^{\partial F_{v_1}}$ in $\Sigma_q F$, for any $q \in F$.

The lemma thus follows. \square

2.3. Extending dist_F -gradient flows.

Since dist_F is concave in $D = A \setminus F$, for each $x \in D$ there is a unique dist_F -gradient curve from x . We call a gradient curve maximal if it is not a proper subset of another gradient curve. Note that any maximal gradient curve has empty intersection with F . We will extend maximal gradient curves to include points in F so that each point in $F - (S, 0)$ is contained in two extended maximal gradient curve. This property plus the simply connectedness of A allow us to choose one such curve for each point in $F - (S, 0)$, such that we can define a “flow”, $\Psi_a^t : (S, a > 0) \rightarrow A$, by $\Psi_a^t((s, a)) = \gamma_a(t)$, where γ is the chosen extended maximal gradient curve at (s, a) , passing any given extended maximal gradient curve at any given (s_0, a) . Our goal is to show that Ψ_a^t is 1-Lipschitz.

To carry out the above, the key is to establish the local separation property for $F - (S, 0)$ (see Lemma 2.11) and the local 1-Lipschitz property for Ψ_a^t .

Before moving on, we need the following two lemmas.

Lemma 2.11. *For any $q \in (S, a) \subset F_u$, $a \neq 0$, $B_F(q, r)$ separates $B(q, r)$, for r small enough, where $B_F(q, r)$ is a closed r -ball in F .*

Proof. For r small enough, we can assume that $B(q, r) \cap F = B_F(q, r)$. By the local version of Perelman’s stability theorem ([24, 4.7]) and Proposition (2.1.1), we can choose r sufficiently small, such that $B(q, r)$ is homeomorphic to a r -ball on $T_q A$, which is homeomorphic to D^n , and $B_F(q, r)$ is homeomorphic to a r -ball on $T_q F$, which is homeomorphic to D^{n-1} . By definition, $B_F(q, r) \cap Fr(B(q, r)) = (Fr(B_F(q, r)))$ in F , which is homeomorphic to S^{n-2} . By considering the double of $B(q, r)$, we get that $B_F(q, r)$ separates $B(q, r)$. \square

Let q, r be as in Lemma 2.11, and let $\bar{U}_q \subset B(q, r)$ be a convex closed neighborhood of q . Then $B_F(q, r)$ separates \bar{U}_q into two components G_{q1}, G_{q2} .

Since dist_F is concave in D , for dist_F -gradient curves $\alpha(t), \beta(t)$, we have that $|\alpha(t)\beta(t)|$ is 1-Lipschitz if there exists a minimal geodesic joining $\alpha(t)$ and $\beta(t)$, for any t , in the domain D . The following property will guarantee the local 1-Lipschitz property for Ψ_a^t .

Lemma 2.12. *For x, y in the interior of the same component say $G_{q1} \setminus \partial \bar{U}_q$ (denoted by G_{b1}°), we have that $[xy] \cap F = \emptyset$.*

In the proof of Lemma 2.12, we need the following theorem about the relationship between the boundary of a convex subset as an Alexandrov space and the set boundary.

Theorem 2.13 ([1]). *Let $C \subset X \in \text{Alex}^m(\kappa)$ be a convex closed subset ($C \in \text{Alex}^m(\kappa)$ with the induced metric). If C has a nonempty interior, then $\partial C = \text{Fr}(C) \cup (C \cap \partial X)$.*

Proof of Lemma 2.12. Based on the local separation and the convexity of F_v , it is easy to check that $\bar{G}_{qi}, i = 1, 2$ are convex. By Theorem 2.13, we have that $F \cap \bar{G}_{qi} \subset \partial \bar{G}_{qi}$. Hence $[xy] \cap F = \emptyset$, because any minimal geodesic between interior points of a convex set does not intersect the boundary of the convex set ([24, 5.2]). \square

We are now in a position for the construction of a nonexpanding map.

For $x \in D$, let $q \in (S, a) \subset F_u, u \in N'$, be a point such that $|xq| = |xF|$. First suppose $a \neq 0$. We will construct extended maximal gradient curves from (S, a) .

For $b \in (S, a)$, let $b_j \in G_{b1}^\circ$, such that $b_j \rightarrow b$. Let $\gamma_b^j : [0, t_0] \rightarrow A$, $t_0 \geq 0$ denote the dist_F -gradient curve from b_j . Since γ_b^j are equi-continuous, after passing to a subsequence, we can suppose that γ_b^j converge to $\gamma_b : [0, t_0] \rightarrow A$ with $\gamma_b(0) = b$. Note that γ_b doesn't depend on the choice of b_j . Indeed, for any $c \in U_b \cap (S, a)$, let γ_c be a similarly constructed curve, namely, let $c_j \in G_{b1}^\circ$, such that $c_j \rightarrow c$. The dist_F -gradient curve from c_j , $\gamma_c^j : [0, t_0] \rightarrow A$, after passing to a subsequence, converge to $\gamma_c : [0, t_0] \rightarrow A$ with $\gamma_c(0) = c$. Then $|\gamma_b(t)\gamma_c(t)| = \lim_{j \rightarrow \infty} |\gamma_b^j(t)\gamma_c^j(t)|, 0 \leq t \leq t_0$. Since $\gamma_b^j(0), \gamma_c^j(0) \in G_{b1}^\circ \subset U_b$, by Lemma 2.12, we have that $[\gamma_b^j(0)\gamma_c^j(0)] \cap F = \emptyset$. Set $\sigma(s) : [0, l] \rightarrow [\gamma_b^j(0)\gamma_c^j(0)]$ with arc-length parametrization. Consider the curves $\sigma_t(s) = \Phi_{\text{dist}_F}^t([\gamma_b^j(0)\gamma_c^j(0)](s))$, where $\Phi_{\text{dist}_F}^t$ is the standard dist_F -gradient flow defined in [30, 2.2]. Let $P_m = \{0 = s_0 < s_1 < \dots < s_m = l, s_i - s_{i-1} = \frac{l}{m}\}$ be a partition of $[0, l]$, with m large enough, such that $\frac{l}{m} \leq \frac{\varepsilon}{10}$, where $\varepsilon = |[\gamma_b^j(0)\gamma_c^j(0)], F|$. If $t \leq \frac{\varepsilon}{4}$, then $[\sigma_t(s_i)\sigma_t(s_{i-1})] \cap F = \emptyset$, thus $|\sigma_t(s_i)\sigma_t(s_{i-1})| \leq |\sigma(s_i)\sigma(s_{i-1})|$. Let $m \rightarrow \infty$, we get that $|\gamma_b^j(t)\gamma_c^j(t)| \leq \text{Length}(\sigma_t) \leq \text{Length}(\sigma_0) = |\gamma_b^j(0)\gamma_c^j(0)|$, where $\text{Length}()$ denotes the length of the curve. If $t > \frac{\varepsilon}{4}$, we can repeat the procedure for $[\sigma_{\frac{\varepsilon}{4}}(s_i)\sigma_{\frac{\varepsilon}{4}}(s_{i-1})]$. Finally, we get that for any $0 \leq t \leq t_0$, $|\gamma_b^j(t)\gamma_c^j(t)| \leq |\gamma_b^j(0)\gamma_c^j(0)|$. When $j \rightarrow \infty$, we get that $|\gamma_b(t)\gamma_c(t)| \leq |bc|$. In particular, we have that if $b = c$, then $\gamma_b = \gamma_c$.

For simplicity, in the following context, we will say that γ_b and γ_c are in the same component with respect to U_b , and we will call γ_b an extended maximal gradient curve, if $\gamma_b - b$ is maximal.

By now we can see that for each point on (S, a) there exist exactly two extended maximal gradient curves. Choose a point $y \in (S, a)$, denote the two extended maximal gradient curves by γ_{y1}, γ_{y0} . For any other point say z , we will denote the two extended maximal gradient curves from z by γ_{z1}, γ_{z0} , such that γ_{zi} are a continuation of γ_{yi} along $[yz]$, i.e., there is a partition of $[yz]$, $P = \{y_0 = y, y_1, \dots, y_k = z\}$, such that $\gamma_{y_j, i}, \gamma_{y_{j+1}, i}$ are in the same component with respect

to U_{y_j} . It doesn't depend on the choice of the partition, since for another partition P_1 , we can consider $P \cup P_1$ to get the independency.

Since $\pi_1((S, a)) = 0$, similarly as the final part of the proof of Key Lemma 0.7, we can see that the denoting doesn't depend on the choice of $[yz]$. Thus we can define two "flows" $\Psi_{ia}^t : (S, a) \rightarrow A$, by $b \rightarrow \gamma_{bi}(t)$, $b \in (S, a)$, and we fix one passing x , which exists, since $[xq]$ is contained in an extended maximal gradient curve, denoted by $\Psi_a^t : (S, a) \rightarrow A$.

Lemma 2.14. *The map $\Psi_a^t : (S, a) \rightarrow A$ is 1-Lipschitz.*

Proof. For $b, c \in (S, a)$, we need to show that $|bc| \geq |\gamma_b(t)\gamma_c(t)|$. Dividing $[bc]$ into small pieces so that the above local 1-Lipschitz property holds, one gets that $|bc| \geq \text{Length}(\Psi_a^t([bc])) \geq |\Psi_a^t(b)\Psi_a^t(c)|$. \square

2.4. Completion of the proof of Theorem A.

Lemma 2.15. *Let $S_x = \Psi_a^{|xq|}((S, a))$. Then $\pi|_{S_x} : S_x \rightarrow S$ is an isometry.*

Proof. Let $i : S \rightarrow (S, a)$ denote the natural isometry. Define a map, $H : (S, a) \times [0, l] \rightarrow (S, a)$, where $l = |xq|$, by $H(x, t) = i \circ \pi \circ \Psi_a^t(x)$. Since $|\Psi_a^t(x), \Psi_a^{t'}(x')| \leq |\Psi_a^t(x), \Psi_a^t(x')| + |\Psi_a^t(x'), \Psi_a^{t'}(x')| \leq |xx'| + |t - t'|$, H is continuous. Since $H(*, 0) = id$, $H(x, l)$ is onto. Recall that given two 1-Lipschitz onto maps between two compact metric spaces, $g : X \rightarrow Y$ and $h : Y \rightarrow X$, then g and h are isometries. Since Ψ_a^t and π are 1-Lipschitz, the desired result follows. \square

If $a = 0$, consider $\Sigma_q A$, from the proof of Lemma 2.8, we get that there are $u, v \in \phi_{[pq]}(N')$ (u may be equal to v), such that $|u \uparrow_q^x| = |v \uparrow_q^x| = \frac{\pi}{2}$. Let $x_i \in [xq], y_i \in ((S, a_i) \cap \pi^{-1}(q)) \subset F_{\phi_{[pq]}^{-1}(u)}$, where $a_i \neq 0$, and $x_i \rightarrow q, y_i \rightarrow q$, as $i \rightarrow \infty$.

Claim 2.16. *There exist $\bar{y}_i \in [x_i y_i] \cap F$, and $\bar{y}_i \notin S$, such that $[x_i \bar{y}_i]^\circ \cap F = \emptyset$.*

Suppose that $\bar{y}_i \in (S, b_i)$ with $b_i \neq 0$. Let $\Psi_{b_i}^t : (S, b_i) \rightarrow A$ denote the "flow" such that, $\Psi_{b_i}^t(\bar{y}_i), t \leq \varepsilon$ (ε small enough), is in the same component (defined after Lemma 2.11) as $[y_i' \bar{y}_i] \subset [x_i \bar{y}_i]$ with y_i', \bar{y}_i close enough, and let $\gamma_i(t) = \Psi_{b_i}^t(\bar{y}_i)$. Then by the construction of $\gamma_i(t)$, one obtains that $|\gamma_i(t), [x_i x](t)| \leq |\bar{y}_i x_i|$, thus $\gamma_i(t) \rightarrow [xq](t)$, for any $0 \leq t \leq |xq|$, as $i \rightarrow \infty$. Hence by passing to a subsequence we can suppose that $\Psi_{b_i}^{|xq|}((S, b_i)) \xrightarrow{\text{isom}} S$ converge to $S_x \xrightarrow{\text{isom}} S$, with $x \in S_x$. Since $\pi|_{\Psi_{b_i}^{|xq|}((S, b_i))}$ is an isometry, so is $\pi|_{S_x}$.

Proof of Claim 2.16. It suffices to show that if $z \in [x_i y_i] \cap F$, then $z \notin S$. Argue by contradiction, suppose $z \in S$. Since $|x_i q| \leq |x_i z|$ and $|y_i q| \leq |y_i z|$, we have that $[x_i q] \cup [q y_i]$ is a minimal geodesic, which is impossible, because by the choice of y_i we have that $\uparrow_q^{x_i} \perp \uparrow_q^{y_i}$. \square

Now we are ready for the proof of Theorem A.

Proof of Theorem A. Case 1: $\mathcal{F} = A$. See the proof following Lemma 1.1.

Case 2: $\mathcal{F} \neq A$. By Lemma 1.5, we can assume that $\pi_1(S) = 0$. For $x \in A$ and for any $\bar{y} \in S$. Let $y = S_x \cap \pi^{-1}(\bar{y})$. Since $\pi|_{S_x}: S_x \rightarrow S$ is an isometry, $|xy| = |\bar{x}\bar{y}|$, and this shows that $\pi: A \rightarrow S$ is a submetry. \square

3. APPLICATION

We will prove Corollary 0.6, and we will show that in Theorem A if $\mathcal{F} = A$, then π is a bundle map. First we recall the following lemma.

Lemma 3.1. *Let $A \in \text{Alex}^4(\kappa)$. Then the following statements are equivalent:*

- (1) *A is topologically nice.*
- (2) *A is topologically regular.*
- (3) *A is a topological manifold.*

Lemma 3.1 is pointed out by Kapovitch in [21], for completeness we include a proof here.

Proof. It is obvious that (1) \Rightarrow (2) and (2) \Rightarrow (3).

(3) \Rightarrow (1): Any manifold point $p \in A$ satisfies that $\pi_1(\Sigma_p A) = 0$ (cf. [41, Theorem 1.1 (2)]), and that $H_*(\Sigma_v \Sigma_p A) = H_*(S^2)$ ([41, Proposition 3.1]). Hence $\Sigma_v \Sigma_p A$ is an S^2 , and therefore $\Sigma_p A$ is a simply connected manifold. By 3-dimension Poincaré conjecture, $\Sigma_p A$ is homeomorphic to S^3 , and thus A is topologically nice. \square

Proof of Corollary 0.6. Argue by contradiction, suppose that $\dim(S) > 0$. By [38, 9.8], $\dim(S) \neq 1, 3$. If $\dim(S) = 2$, we may assume that $q \in A$ such that the curvature are positive in $B(q, r)$. Then the modified Busemann function $-\exp^f$ is strictly concave ([8]), and thus $\pi|_{B(q, r)}: B(q, r) \rightarrow B(\pi(q), r)$ is strictly distance decreasing; a contradiction to Theorem A. \square

Corollary 3.2. *Let the assumptions be as Theorem A. Suppose $\mathcal{F} = A$. Then $\pi: A \rightarrow S$ is a bundle map with fiber \mathbb{R}^2 .*

Note that Corollary 3.2 doesn't hold, if one removes the condition of topologically nice see [42, 14.8].

In the proof, we shall apply the following theorem which is a sufficient condition for a bundle map:

Theorem 3.3 ([33]). *Let X, X_0 be two metric spaces and let $f: X \rightarrow X_0$ be a continuous onto map. Suppose that for any $p \in X_0$ and $\varepsilon > 0$ there exists a $\delta(p, \varepsilon) > 0$ such that for any $q \in B(p, \delta)$, there is a homeomorphism $h: f^{-1}(p) \rightarrow f^{-1}(q)$, with $|h(x)x| \leq \varepsilon$, for any $x \in f^{-1}(p)$. If the fiber F is locally compact and separable, and the homeomorphism group of F (with some natural topology) is locally path connected, then F is a Serre fibration. If in addition, X_0 is finite dimensional ANR, then f is a locally trivial bundle map.*

Proof of Corollary 3.2. Since $\mathcal{F} = A$, for any $p, q \in S$ and a fixed $[pq]$, there is a homeomorphism $h: \pi^{-1}(p) \rightarrow \pi^{-1}(q)$, such that $[xp], [pq], [qh(x)]$ determine a unique flat rectangle (Lemma (1.1.2) and Proposition (2.1.3)). Then $|h(x)x| = |pq|$.

Since A is topologically nice and π is a submetry, if $p \in S$ is a regular point (i.e., $T_p S$ is isometric to \mathbb{R}^{n-2}), then $\pi^{-1}(p)$ is a topological manifold ([32, Theorem D], note that the proof is local, and thus apply to non-compact cases). Since $\pi : \pi^{-1}(p) \rightarrow p$ is a deformation retraction, $\pi^{-1}(p) \stackrel{\text{homeo}}{\simeq} \mathbb{R}^2$.

And by [11, 7.3], we know that the homeomorphism group $\text{homeo}(\pi^{-1}(p))$ is locally path connected with the topology in Theorem 3.3. By now we are able to apply Theorem 3.3 to conclude that π is a bundle map. \square

Remark 3.4. Inspired by [40], it seems that for Case 2 of Theorem A, when $\pi_1(A) = 0$, A isometrically splits.

4. STRUCTURE OF SPACE OF DIRECTIONS

Our main efforts in this section is to prove Proposition 2.1, and thus complete the proof of Theorem A. We point out that Theorem 4.4, which classifies certain isometric class in $\text{Alex}^n(1)$, may have independent interest.

First we recall the following:

Definition 4.1 ([3]). Let $X, Y \in \text{Alex}(1)$. The join of X and Y , $X * Y$, is defined by $X \times Y \times [0, \frac{\pi}{2}] / \sim$, where $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, \frac{\pi}{2}) \sim (x_2, y, \frac{\pi}{2})$, with the metric: $\cos d((x_1, y_1, t), (x_2, y_2, s)) = \cos t \cos s \cos(|x_1 x_2|) + \sin t \sin s \cos(|y_1 y_2|)$.

Then $X * Y \in \text{Alex}(1)$ and $C(X * Y) = C(X) \times C(Y)$ ([3]).

Example 4.2. $S^m * S^n = S^{m+n+1}$, where all the spheres are with the standard metric with constant curvature 1.

Remark 4.3. For the convenience of following use, we will make the following conventions:

- (1) Let 0-dim Alexandrov space with curvature bounded below by 1 without boundary be a space including two points with distance π , one point is regraded as with boundary.
- (2) If a subspace with restricted metric is isometric to an Alexandrov space, we also say that it is convex, although when the dimension is 0, there may not be a minimal geodesic in the subspace joining two given points.
- (3) When we say that two metric spaces are equal, we always mean metrically, except otherwise stated. If there is no confusion, we will not mention the metric.

Let $A \in \text{Alex}^n(1)$ and let $C \subset A$ be a closed convex subset without boundary. Let $\hat{\Sigma}_p C = \{v \in \Sigma_p A \mid |v \Sigma_p C| \geq \frac{\pi}{2}\}$, and let $\hat{T}_p C = C(\hat{\Sigma}_p C)$. Observe that if A and C are Riemannian manifolds, then $T_p A$ isometrically splits, i.e., $T_p A = T_p C \times \hat{T}_p C$ or equivalently $\Sigma_p A$ is isometric to the join of $\Sigma_p C$ and $\hat{\Sigma}_p C$ ([3]). In Alexandrov geometry, such property doesn't hold.

Observe that for $\Sigma \in \text{Alex}^n(1)$ and $\Sigma_0, \Sigma_1 \subset \Sigma$ convex closed subsets, if $\Sigma = \Sigma_0 * \Sigma_1$, then Σ_0, Σ_1 satisfy the following conditions:

- (1) For every $v \in \Sigma$, $v \in [v_0 v_1]$, for some $v_0 \in \Sigma_0, v_1 \in \Sigma_1$.
- (2) $\dim(\Sigma_0) + \dim(\Sigma_1) + 1 = n$. (\dim denotes the Hausdorff dimension.)

In the following situation, conditions (1) and (2) are indeed sufficient conditions for join.

Theorem 4.4. *Let $\Sigma \in \text{Alex}^n(1)$, let $\Sigma_0 \subset \Sigma$ be a closed convex subset without boundary, and let $\Sigma_1 = \{v \in \Sigma \mid |v\Sigma_0| \geq \frac{\pi}{2}\}$. Assume that $\Sigma_1 \neq \emptyset$ and Σ_0, Σ_1 satisfy (1) and (2). If Σ is topologically nice and homeomorphic to a sphere, then $\Sigma = \Sigma_0 * \Sigma_1$ and $\Sigma_i, i = 0, 1$, are topologically nice and homeomorphic to spheres.*

Theorem 4.4 was related to and inspired by the following Conjecture 4.5 made by Yamaguchi [42, 14.6].

Conjecture 4.5. *Let $A \in \text{Alex}^n(0)$ be noncompact with a soul S . If $p \in (C_i - \partial C_i)$ is a topologically regular point, then $T_p X$ is isometric to the product $T_p C_i \times K$, where C_i are as in 1.1, and K is a Euclidean cone.*

Yamaguchi proved Conjecture 4.5 for $n = 4$ ([42, Theorem 14.5]); where topologically nice and topologically regular are equivalent.

Using Theorem 4.4, with additional argument we can get the following classification result, which can imply Proposition 2.1.

Theorem 4.6. *Let $\Sigma \in \text{Alex}^n(1), n \geq 2$. Let $\Sigma_0 \subset \Sigma$ be a convex closed subset without boundary with dimension $n - 2$, and let $\Sigma_1 = \{u \in \Sigma \mid |u\Sigma_0| \geq \frac{\pi}{2}\}$. Suppose that $\Sigma_1 \neq \emptyset$, Σ is homeomorphic to a sphere and topologically nice, and Σ satisfies that for any $u \in \Sigma$, $|u\Sigma_0| \leq \frac{\pi}{2}$ and $|u\Sigma_1| \leq \frac{\pi}{2}$. Then Σ_1 is convex, Σ_0 is homeomorphic to S^{n-2} and topologically nice, and $[\Sigma_0\Sigma_1] = \Sigma_0 * \Sigma_1$, where $[\Sigma_0\Sigma_1]$ is with the restricted metric. Explicitly, we get the following classification:*

- 1) *If $\Sigma_1 = S^1(r)$ with $r \leq 1$, then $\Sigma = \Sigma_0 * \Sigma_1$.*
- 2) *If $\Sigma_1 = \{v_1, v_2\}$ with $|v_1 v_2| = \pi$, then $\Sigma = \Sigma_1 * \Sigma_1^\perp$.*
- 3) *If $\Sigma_1 = \{v\}$, then $\Sigma = D(\Sigma_1 * \hat{\Sigma}_1)$; the double of half suspension of $\hat{\Sigma}_1 = \{u \in \Sigma \mid |u\Sigma_1| \geq \frac{\pi}{2}\}$.*
- 4) *If $\Sigma_1 = [ab]$, then $\Sigma = (a * \hat{\Sigma}_1 \cup_{\hat{\Sigma}_1} b * \hat{\Sigma}_1) \cup_{\partial} (\Sigma_1 * \Sigma_0)$ with the gluing metric ([2, Definition 3.1.12]).*

In the rest of this section, we will prove Theorem 4.4 and Theorem 4.6. First we will present some preparations.

Lemma 4.7. *Let $\Sigma \in \text{Alex}^n(1)$ and let $\Sigma_1, \Sigma_0 \subset \Sigma$ be two convex closed subsets with dimension k, l respectively. Suppose that Σ satisfies: for any $v_0, v_1 \in \Sigma_0, \Sigma_1$ respectively, $|v_0 v_1| = \frac{\pi}{2}$. Then $n \geq l + k + 1$.*

Proof. Let $C_i \subset \Sigma_i \setminus \partial \Sigma_i, i = 1, 0$, be two closed convex subsets. Note that $[C_1 C_0]$ with the restricted metric is a closed subset of Σ . It suffices to construct a nonexpanding map from $[C_1 C_0]$ to $C_1 * C_0$. If so, $\dim(\Sigma) \geq \dim([C_1 C_0]) \geq \dim(C_1 * C_0) = k + l + 1$, where the last identity is because, by definition, we have that $\dim(C(C_0 * C_1)) = \dim(C(C_0) \times C(C_1)) = k + l + 2$.

First we claim that $[x_1 y_1] \cap [x_2 y_2] \subset \{x_1, y_1, x_2, y_2\}$, for $x_1, x_2 \in C_1, y_1, y_2 \in C_0$. Thus for $p \in [C_1 C_0]$, p can be uniquely written as $([xy], s)$, where $x \in C_1, y \in C_0, p \in [xy]$ and $s = |px|$. Then we can construct a map, $g : [C_1 C_0] \rightarrow C_1 * C_0$, by $g|_{C_i} = \text{id}, i = 0, 1$ and $g([pq], s) = ([pq], s)$ (the unique point on the unique minimal geodesic $[pq]$ with distance s from p), for $p \in C_1, q \in C_0$.

Next we will show that g is nonexpanding. In order to do so we need the following fact:

For $p \in C_1$ and $q \in C_0$, $|pq| = |p\Sigma_0|$. By Lemma 1.2, $|\uparrow_q^p v| = \frac{\pi}{2}$, for any $v \in \Sigma_q \Sigma_0$. Hence $\tilde{\angle}(p, q, r) = \angle(p, q, r) = \frac{\pi}{2}$, for any $r \in \Sigma_0$. By Lemma 2.6, there is a convex triangle isometric to the corresponding one on space form.

For $x, y \in [C_1 C_0]$, by the property of $[C_1 C_0]$, we have that $x \in [p_1 p_0]$, $y \in [q_1 q_0]$. Let $|xp_1| = s$ and $|yq_1| = t$. Then by the above fact, $\cos(|p_1 y|) = \cos(|p_1 q_1|) \cos t$, $\cos(|p_0 y|) = \cos(|p_0 q_0|) \sin t$. By the monotonicity of angle, $|xy| \geq \cos s \cos t \cos(|p_1 q_1|) + \sin s \sin t \cos(|p_0 q_0|) = |g(x)g(y)|$, which shows that g is nonexpanding.

Finally, we will verify the claim. It suffices to show that if $x_1 \neq x_2, y_1 \neq y_2$, then $[x_1 y_1] \cap [x_2 y_2] = \emptyset$. Suppose that $[x_1 y_1] \cap [x_2 y_2] = z$. Hence by the above fact, $[z y_1] \subset$ the totally geodesic triangle bounded by $\{x_2, y_1, y_2\}$, which is isometric to the corresponding triangle on $S^2(1)$, or the geodesic will branch. Thus $\angle(x_1, y_1, y_2) < \frac{\pi}{2}$. This is a contradiction. \square

Remark 4.8. Let $\Sigma \in \text{Alex}^n(1)$, and let $\Sigma_0 \subset \Sigma$ be a convex closed subset. Suppose $\Sigma_1 = \{v \in \Sigma \mid |v\Sigma_0| \geq \frac{\pi}{2}\} \neq \emptyset$. If $\partial\Sigma_0 = \emptyset$ or $\partial\Sigma_1 = \emptyset$, then by Lemma 1.2, Σ satisfies the conditions of Lemma 4.7. These are the cases usually used in the following context.

4.1. Proof of Theorem 4.4.

Sketch of the proof: we will prove Theorem 4.4 by induction on the dimension of Σ . And we will prove the inductive step according to the different situations of the dimension of Σ_0 and Σ_1 . Except two simple cases, we will first show that for each point in Σ_1 and each point in Σ_0 , there exist exactly m minimal geodesics joining them for some $m > 0$. Then argue by contradiction, we can get that $m = 1$, using this we can derive that Σ is a join.

Proof of Theorem 4.4. Denote $\dim(\Sigma_1)$ by I and $\dim(\Sigma_0)$ by J . First we will prove the following property.

Sublemma 4.9. *We have that Σ_1 is convex.*

Proof. For $p, q \in \Sigma_1$, when $|pq| < \pi$, by triangle comparison we have that for any $x \in \Sigma_0$, $|x, [pq]| \geq \frac{\pi}{2}$. Hence $[pq] \subset \Sigma_1$. When $|pq| = \pi$, if there is no other points in Σ_1 , then the sublemma holds. If there is $x \in \Sigma_1 - \{p, q\}$, one can easily see that $|px| + |xq| = \pi = |pq|$, thus $[px], [xq] \subset \Sigma_1$, and $[px] \cup [xq]$ is a minimal geodesic which lies in Σ_1 . The sublemma thus follows. \square

We then proceed the proof by induction on n . For $n = 1$, i.e., $I = 0$ and $J = 0$, clearly the theorem holds. Suppose $n - 1$ the theorem holds.

Now we will prove the inductive step according to different situations of I and J :

If $J = 0$, i.e., $\Sigma_0 = \{v_1, v_2\}$ with $|v_1 v_2| = \pi$, then $\Sigma = \Sigma_0 * \Sigma_1$. Thus $\Sigma_{v_1} = \Sigma_1$. Because Σ is topologically nice, Σ_1 is homeomorphic to S^{n-1} and topologically nice.

Hence in the following we can assume that $J > 0$.

Next we want to show that $\partial\Sigma_1 = \emptyset$. In order to apply inductive assumptions to $\Sigma_p\Sigma$, for $p \in \Sigma_0$, it suffices to check that $\Sigma_p\Sigma_0, (\Sigma_p\Sigma_0)^\perp$ satisfy conditions (1) and (2). For condition (1):

Sublemma 4.10. *For any $p \in \Sigma_0$ and for any $\bar{v} \in \Sigma_p\Sigma, \bar{v} \in [\bar{v}_1\bar{v}_0] \subset [\uparrow_p^{\Sigma_1} \Sigma_p\Sigma_0]$.*

Sublemma 4.10 implies that for $p \in \Sigma_0, (\Sigma_p\Sigma_0)^\perp = \uparrow_p^{\Sigma_1}$ with the restricted metric in Σ_p .

Proof of Sublemma 4.10. Observe that by Lemma 1.2, we have that for every $p \in \Sigma_1$ and every $q \in \Sigma_0, |pq| = \frac{\pi}{2}$. Hence for every $r \in \Sigma_0, \angle(p, q, r) = \tilde{\angle}(p, q, r)$. By Lemma 2.6, we have that there exists a totally geodesic triangle determined by $[pq]$ and $[qr]$, which is isometric to a geodesic triangle in $S^2(1)$.

Let $\bar{v} \in (\Sigma_p\Sigma)'$. Then $\bar{v} = \uparrow_p^x$ for some $x \in \Sigma$. By perturbing x along $[px]$, we can suppose that $[px]$ is unique. From condition (1), $x \in [x_1x_0]$, for some $x_i \in \Sigma_i$. Therefore there exists a totally geodesic triangle T determined by $[x_0p], [x_1p]$, which is isometric to a geodesic triangle in $S^2(1)$, and by the uniqueness, we get that $[px] \subset T$. Thus $\bar{v} \in [\uparrow_p^{x_1} \uparrow_p^{x_0}]$. Since $(\Sigma_p\Sigma)'$ is dense in $\Sigma_p\Sigma$, we get the Sublemma. \square

For condition (2): from the construction, we see that, $\psi_1 : \uparrow_p^{\Sigma_1} \rightarrow \Sigma_1, \psi_1(\uparrow_p^x) = x, x \in \Sigma_1$, is a submetry (since ψ_1 has the horizontal lifting property, as can be seen in the proof of Sublemma 4.10). Thus the map ψ_1 is 1-Lipschitz. Therefore $\dim(\uparrow_p^{\Sigma_1}) \geq \dim(\Sigma_1)$. Hence on one hand, we have

$$\dim(\Sigma_p\Sigma) = \dim(\Sigma_0) - 1 + \dim(\Sigma_1) + 1 \leq \dim(\Sigma_p\Sigma_0) + \dim(\uparrow_p^{\Sigma_1}) + 1.$$

On the other hand, by Lemma 4.7, we have the opposite inequality. Thus

$$\dim(\Sigma_p\Sigma) = \dim(\Sigma_p\Sigma_0) + \dim(\uparrow_p^{\Sigma_1}) + 1.$$

By now, we obtain that $\Sigma_p\Sigma, p \in \Sigma_0$ satisfies the inductive assumptions. Hence for every $p \in \Sigma_0, \Sigma_p\Sigma = \Sigma_p\Sigma_0 * (\Sigma_p\Sigma_0)^\perp$, and $\Sigma_p\Sigma_0$ is homeomorphic to a sphere. Thus Σ_0 is a topological manifold.

For the following use, we recall Alexander duality [19]: Let $K \subset S^n$ be a compact, locally contractible, nonempty, proper subspace. Then $\tilde{H}_i(S^n - K; \mathbb{Z}) = \tilde{H}^{n-i-1}(K; \mathbb{Z})$, for all i , where \tilde{H}_i, \tilde{H}^i denote the reduced homology and cohomology.

Now we are ready to show that $\partial\Sigma_1 = \emptyset$. Argue by contradiction, assume $\partial\Sigma_1 \neq \emptyset$. Then Σ_1 is contractible, since Σ_1 has positive curvature (cf. [24]). From condition (1), we see that dist_{Σ_1} has no critical points in $\Sigma - \Sigma_1 \cup \Sigma_0$. Thus we have that there is a deformation retraction from $\Sigma - \Sigma_1$ to Σ_0 . Therefore $\tilde{H}^{n-1-i}(\Sigma_1) = 0$. On one hand by Alexander duality $\tilde{H}_i(S^n - \Sigma_1) = \tilde{H}_i(\Sigma_0) = \tilde{H}^{n-1-i}(\Sigma_1) = 0$, for any $i \geq 0$. On the other hand, since Σ_0 is a topological manifold, it follows that if Σ_0 is orientable, $\tilde{H}_n(\Sigma_0) \neq 0$, and if Σ_0 is not orientable, $\tilde{H}_{n-1}(\Sigma_0) \neq 0$. This is a contradiction. Therefore $\partial\Sigma_1 = \emptyset$.

Now we come to the second case:

If $I = 0$, we have that $\Sigma_1 = \{w_1, w_2\}$ with $|w_1 w_2| = \pi$. Then $\Sigma = \Sigma_0 * \Sigma_1$. Because Σ is topologically nice, Σ_0 is homeomorphic to S^{n-1} and topologically nice.

Hence in the following we can suppose that $I > 0$. First we will show that for any $x_i \in \Sigma_i, i = 0, 1$, there are m minimal geodesics joining x_0 and x_1 , for some $m > 0$.

Similarly, we have that for every $x \in \Sigma_1$,

$$\dim(\Sigma_x \Sigma) = \dim(\Sigma_x \Sigma_1) + \dim(\uparrow_x^{\Sigma_0}) + 1.$$

Hence $\dim(\uparrow_x^{\Sigma_0}) = \dim(\Sigma_0)$.

Since $\partial \Sigma_1 = \emptyset$, we can apply Sublemma 4.10 to $\Sigma_x \Sigma$, $x \in \Sigma_1$. We get that $\Sigma_x \Sigma_1, (\Sigma_x \Sigma_1)^\perp$ satisfy condition (1) and $\uparrow_x^{\Sigma_0} = (\Sigma_x \Sigma_1)^\perp$. Hence $\uparrow_x^{\Sigma_0}$ is convex, and similarly, $\psi_0 : \uparrow_x^{\Sigma_0} \rightarrow \Sigma_0, \psi_0(\uparrow_x^p) = p, p \in \Sigma_0$, is a submetry. Thus for a (J, δ) -burst point (see [3, 5.2]) $y \in \Sigma_0$, there is a neighborhood U_y of y , such that $\psi_0^{-1}(U_y) \stackrel{\text{homeo}}{\simeq} U_y \times F_0$, where F_0 is a 0-dim MCS-space ([24]). Hence F_0 is a collection of discrete points. Since $\uparrow_x^{\Sigma_0}$ is compact, $\#\{\psi_0^{-1}(y)\} < \infty$, say m . In the following, we will show that m is independent of the choice of y .

Sublemma 4.11. *For every $x \in \Sigma_1$ and every $p, q \in \Sigma_0$, $\#\{\uparrow_x^p\} = \#\{\uparrow_x^q\}$.*

Proof. Since for a fixed minimal geodesic $[pq]$ and for any $[xp]$, there is a totally geodesic triangle with two sides $[pq], [xp]$, which is isometric to a geodesic triangle in $S^2(1)$, there is a corresponding $[xq]$. This determines a map from \uparrow_x^p to \uparrow_x^q , which is 1-1, onto, or there will be a contradiction to the join of Σ_p or Σ_q , and thus the sublemma follows. \square

Then we want to prove the following:

Sublemma 4.12. *The map ψ_0 is an m -fold locally isometric covering map.*

Proof. Let $\psi_0^{-1}(p) = \cup_{i=1, \dots, m} \{p_i\}$ and $\varepsilon = \min_{1 \leq i, j \leq m} \{|p_i p_j|\}$. Then $B(p_i, \frac{\varepsilon}{5}) \cap B(p_j, \frac{\varepsilon}{5}) = \emptyset$. Since if there is $x \in B(p_i, \frac{\varepsilon}{5}) \cap B(p_j, \frac{\varepsilon}{5})$, then $|p_i p_j| \leq |xp_i| + |xp_j| \leq \frac{\varepsilon}{2}$, a contradiction. Hence $\psi_0^{-1}(B(p, \frac{\varepsilon}{5})) = B(\psi_0^{-1}(p), \frac{\varepsilon}{5}) = B(\cup \{p_i\}, \frac{\varepsilon}{5}) = \cup B(p_i, \frac{\varepsilon}{5})$.

Next we will show that $B(p, \frac{\varepsilon}{5})$ is isometric to $B(p_i, \frac{\varepsilon}{5})$, for any $1 \leq i \leq m$. For any $\bar{x} \in B(p, \frac{\varepsilon}{5})$, there exists a unique $x \in B(p_i, \frac{\varepsilon}{5})$ such that $|p_i x| = |p \bar{x}|$, or will contradict to Sublemma 4.11. For $\bar{y} \in B(p, \frac{\varepsilon}{5})$, let $y \in B(p_i, \frac{\varepsilon}{5})$ such that $|yx| = |\bar{x} \bar{y}|$. Then $|p_i y| = |p \bar{y}|$, or $|p_i p_j| \leq \frac{4\varepsilon}{5}$, a contradiction. Thus we get the sublemma. \square

Since we have showed that for any $x \in \Sigma_1$, $\Sigma_x \Sigma$ satisfies the conditions of the theorem, by induction assumption, $\Sigma_x \Sigma = \Sigma_x \Sigma_1 * \uparrow_x^{\Sigma_0}$, and $\Sigma_x \Sigma_1, \uparrow_x^{\Sigma_0}$ are homeomorphic to spheres.

Hence $\Sigma_0 \stackrel{\text{homeo}}{\simeq} S^J / \Gamma_0$. Similarly, we get that for any $p \in \Sigma_0, \psi_1 : \uparrow_p^{\Sigma_1} \rightarrow \Sigma_1$ is an m -fold covering map, and therefore $\Sigma_1 \stackrel{\text{homeo}}{\simeq} S^I / \Gamma_1$. By now we can see that for any $x_i \in \Sigma_i$, there are m minimal geodesics joining x_0 and x_1 .

Next, we want to show that $m = 1$.

Sublemma 4.13. *Let $X_i = \{v \mid |v\Sigma_i| \leq \frac{\pi}{4}\}$, $i = 0, 1$. Then X_i are locally trivial bundles over Σ_i .*

Proof. Without loss of generality, we can suppose that $i = 0$. First observe that there is a natural map, $g : X_0 \rightarrow \Sigma_0$, defined in the following way: for any $y \in X_0$, by condition (1), we have that $y \in [y_1 y_0]$, for some $y_i \in \Sigma_i$, set $g(y) = y_0$.

For $p \in \Sigma_0$, let U be a locally isometric neighborhood of p as in Sublemma 4.12. Then U satisfies that the covering map $\psi_0 : \uparrow_x^{\Sigma_0} \rightarrow \Sigma_0$, for any $x \in \Sigma_1$, when restricted to each component of $\psi_0^{-1}(U)$, is an isometry. Indeed, if Σ_0 is an S^1 , then nothing need to say. If Σ_0 is not an S^1 , let Y_1, Y_2 be two universal covering spaces of Σ_0 , with covering maps c_1, c_2 . There exists an isometry, $g : Y_1 \rightarrow Y_2$, which is fiber preserving, i.e., $c_2 \circ g = c_1$. Then we can get a neighborhood V , such that if c_1 , when restricted to $c_1^{-1}(V)$, is an isometry, so is c_2 .

Now we can define a map, $\eta : g^{-1}(U) \rightarrow U \times F$, where $F = g^{-1}(p)$, in the following way: for $z \in g^{-1}(U)$, then $z \in [z_1 z_0]$ for some $z_i \in \Sigma_i$, set $\eta(z) = (z_0, ([z_1 p], |zz_1|))$, where $([z_1 p], |zz_1|)$ represents a point on the geodesic $[z_1 p]$ with distance $|zz_1|$ from z_1 , and $[z_1 p]$ is the geodesic such that $[\uparrow_{z_1}^p \uparrow_{z_1}^{z_0}]$ is the horizontal lifting of $[pz_0]$, i.e., $|\uparrow_{z_1}^p \uparrow_{z_1}^{z_0}| = |pz_0|$, which is unique, by the choice of U . Then η is injective and onto. Next we will check that η is continuous. For points $z_i \in [z_i^1 z_i^0]$, $z_i \rightarrow y \in [y_1 y_0]$, since $|py_0| \leq |\uparrow_{y_1}^p \uparrow_{y_1}^{y_0}| \leq \liminf |\uparrow_{z_i^1}^p \uparrow_{z_i^1}^{z_i^0}| = \liminf |pz_i^0| = |py_0|$, we can get the continuity of η . On the other hand, we can see that the inverse of η , $\bar{\eta} : U \times F \rightarrow g^{-1}(U)$, is defined by $\bar{\eta}((x_0, f)) = ([f_1 x_0], |ff_1|)$, where $f \in [pf_1]$, and $[\uparrow_{f_1}^{x_0} \uparrow_{f_1}^p]$ is the horizontal lifting of $[x_0 p]$. Likewise, we can see that $\bar{\eta}$ is injective, onto and continuous. Hence η is a homeomorphism, and obviously $p_2 \circ \eta = g$, where $p_2 : U \times F \rightarrow F$ is the projection to the second factor. Thus we get the Sublemma. \square

Thus X_1 is D^{J+1} -bundle over S^I/Γ_1 and X_0 is D^{I+1} -bundle over S^J/Γ_0 .

If $I = 1$ and $J = 1$ (cf. [42, Theorem 14.5]), Σ is glued by two solid tori (since by [19] we know that Klein bottle can not be embedded into S^3). Hence Σ is a Lens space with $\pi_1(\Sigma) = \mathbb{Z}_m$. Therefore $m = 1$.

If $I = 1$ and $J > 1$, by the long exact sequences:

$$\begin{aligned} & \rightarrow \pi_1(S^J) \rightarrow \pi_1(\partial X_1) \rightarrow \pi_1(S^1) \rightarrow 0, \\ & \rightarrow \pi_1(S^1) \rightarrow \pi_1(\partial X_0) \rightarrow \pi_1(S^J/\Gamma_0) \rightarrow 0, \end{aligned}$$

we have that $\pi_1(\partial X_1) = \mathbb{Z}$ and Γ_0 is a quotient group of \mathbb{Z} . Hence $\Gamma_0 = \mathbb{Z}_m$. By Alexander duality, we have that $\mathbb{Z}_m = \tilde{H}_1(S^J/\Gamma_0; \mathbb{Z}) = \tilde{H}_1(S^n \setminus S^1; \mathbb{Z}) = \tilde{H}^{n-1-1}(S^1; \mathbb{Z}) = 0$, where the last identity is because $n > 3$. Thus $m = 1$. Similarly for $I > 1$ and $J = 1$.

If $I > 1$ and $J > 1$, denote $\tilde{X}_i = \psi_i^*(X_i)$; the pull back bundle by the covering map ψ_i . Then \tilde{X}_i is an m -fold cover of X_i . The covering map is denoted by $\phi_i, i = 0, 1$. Since $\pi_1(\tilde{X}_i) = 0$, \tilde{X}_i is the universal cover of X_i .

Because $\partial\tilde{X}_1$ and $\partial\tilde{X}_0$ are both universal coverings of $\partial X_1 = \partial X_0$, there exists a homeomorphism $h_0 : \partial\tilde{X}_1 \rightarrow \partial\tilde{X}_0$, such that $\phi_1 = \phi_0 \circ h_0$. Set $\tilde{X} = \tilde{X}_1 \cup \tilde{X}_0 / \sim$ (topologically), where $x \sim h_0(x)$, when $x \in \partial\tilde{X}_1$. We will show that \tilde{X} is an m -fold cover of X .

Let $h : \tilde{X} \rightarrow X$ be defined by:

$$h(x) = \begin{cases} \phi_1(x), & x \in \tilde{X}_1, \\ \phi_0(x), & x \in \tilde{X}_0. \end{cases}$$

If $x \in \partial\tilde{X}_1$, $h(x) = \phi_1(x) = \phi_0(h_0(x))$. Thus h is well defined and continuous. Next we will check that h is a covering map.

Since for $x \in X_1 \cap X_0$, there exists a small neighborhood U of x , denote $U_i = X_i \cap U$, such that for the closure \bar{U}_i , we have that $\phi_i^{-1}(\bar{U}_i) = \cup_{j=1, \dots, m} \bar{U}_{ij}, i = 0, 1$, with $\phi_i|_{\bar{U}_{ij}}$ are homeomorphisms. Thus $h^{-1}(\bar{U}) = \cup_{j=1, \dots, m} (\bar{U}_{1j} \cup_{h_0} \bar{U}_{0j'})$. And it is clear that $h|_{\bar{U}_{1j} \cup_{h_0} \bar{U}_{0j'}}$ is bijective and onto, therefore a homeomorphism (since $\bar{U}_{1j} \cup_{h_0} \bar{U}_{0j'}$ is compact). Thus $h|_{U_{1j} \cup_{h_0} U_{0j'}}$ is a homeomorphism. It follows that \tilde{X} is an m -fold cover of Σ , and therefore $m = 1$.

Finally, we will show that $\Sigma = \Sigma_0 * \Sigma_1$. As can be seen in the proof of Lemma 4.7, we have that for $p \in \Sigma$, p can be uniquely written as $([xy], s)$, where $x \in \Sigma_1$, $y \in \Sigma_0$ and $s = |px|$. By now we can construct a map, $\xi : \Sigma \rightarrow \Sigma_0 * \Sigma_1$, such that $\xi|_{\Sigma_i}, i = 0, 1$ are isometries, and maps $u = ([xy], s)$ to $([\xi(x)\xi(y)], s)$, where $s = |\xi(u)x|$. For every $p_1 = ([x_1y_1], s)$, $p_2 = ([x_2y_2], t) \in \Sigma$, since $|x_1p_2| = |\xi(x_1)\xi(p_2)|$ and $|y_1p_2| = |\xi(y_1)\xi(p_2)|$, by triangle comparison, we get that $|p_1p_2| \geq |\xi(p_1)\xi(p_2)|$. Hence $\angle(\uparrow_p^{z_1}, \uparrow_p^{z_2}) \geq \angle(\uparrow_{\xi(p)}^{\xi(z_1)}, \uparrow_{\xi(p)}^{\xi(z_2)})$, for $p \in \Sigma_1$ and $z_i \in \Sigma$, that is, the induced map, $\xi_* : \Sigma_p \Sigma \rightarrow \Sigma_{\xi(p)}(\Sigma_0 * \Sigma_1)$, is 1-Lipschitz. Since $\Sigma_p \Sigma$ is isometric to $\Sigma_{\xi(p)}(\Sigma_0 * \Sigma_1)$, we can see that ξ_* is an isometry. Specially, $\angle(\uparrow_p^{p_1}, \uparrow_p^{p_2}) = \angle(\uparrow_{\xi(p)}^{\xi(p_1)}, \uparrow_{\xi(p)}^{\xi(p_2)})$. Then by hinge comparison, we have that $|p_1p_2| \leq |\xi(p_1)\xi(p_2)|$. Thus we get that ξ is an isometry. \square

4.2. Proof of Theorem 4.6.

Before we focus on the proof of Theorem 4.6, we collect some lemmas which will be used in the proof of Theorem 4.6. We begin with the following very simple lemma:

Lemma 4.14. *Let $X \in \text{Alex}^n(\kappa)$, $n \geq 2$ and let $C \subset X$ be a closed locally convex subset with $\dim(C) = n$. If $\partial C \neq \emptyset$, suppose that $C \cap \partial X = \partial C$, then $C = X$.*

Proof. First suppose that $\partial C = \emptyset$. We will proceed by induction, if $n = 1$, we can see that the lemma holds. For the inductive step, argue by contradiction.

Suppose that there is $x \in X \setminus C$, let $y \in C$ be a point such that $|xy| = |xC|$. By induction, we have that $\uparrow_y^x \in \Sigma_y C$. This is a contradiction.

If $\partial C \neq \emptyset$, by considering the double of X , we can get the desired result. \square

Lemma 4.15. *Let the assumptions be as in Theorem 4.6. If $\Sigma_1 = S^1(r)$ with $r \leq 1$, then for any $v \in \Sigma$, there are $v_i \in \Sigma_i, i = 0, 1$, such that $v \in [v_0 v_1]$.*

Proof. For $v \in \Sigma$, let $w \in \Sigma_0$ such that $|vw| = |v\Sigma_0|$. Then by Lemma 1.2, we have that $\uparrow_w^v \in (\Sigma_w \Sigma_0)^\perp$. Since $\uparrow_w^{S^1} = (\Sigma_w \Sigma_0)^\perp = S^1$, the result follows. \square

Lemma 4.16. *Let $A \in \text{Alex}^n(1)$, $\partial A \neq \emptyset$. Then A can't contain a convex closed subset without boundary with positive dimension whose intersection with ∂A is empty.*

Proof. Argue by contradiction. Suppose that there exists a convex closed subset C without boundary with positive dimension such that $C \cap \partial A$ is empty. Let $h = \text{dist}_{\partial A}$ and let $p \in C$ be a point such that $h(p) = \min_{x \in C} \{h(x)\}$. By the first variation formulae, $|\uparrow_p^{\partial A} \Sigma_p C| \geq \frac{\pi}{2}$. By Lemma 1.2, $|\uparrow_p^{\partial A} v| = \frac{\pi}{2}$, for any $v \in \Sigma_p C$.

Then for any $x \in C$, we have that $0 = d_p h(\uparrow_p^x) \geq \frac{h(x) - h(p)}{|px|} \Rightarrow h(p) \geq h(x)$.

Hence $h(x) = \text{constant}$, for any $x \in C$. Let $\gamma \subset C$ be a minimal geodesic. By Theorem 0.5, there exists a flat rectangle, a contradiction to $A \in \text{Alex}^n(1)$. \square

Lemma 4.17. *Let $\Sigma \in \text{Alex}^n(1)$. Suppose that Σ is homeomorphic to a sphere and topologically nice. Let $\Sigma_0 \subset \Sigma$ be a convex closed subset with dimension $n-1$, without boundary. Then Σ is homeomorphic to $S(\Sigma_0)$, and Σ_0 is homeomorphic to a sphere and topologically nice.*

In the proof we will use the following lemma.

Lemma 4.18 ([24, 6.2]). *Let $C \in \text{Alex}^n(1)$ with $\partial C \neq \emptyset$. Let $x \in C$ be the unique point with maximal distance from ∂C . Then $(C, \partial C) \xrightarrow{\text{homeo}} (\bar{C}(\Sigma_x C), \Sigma_x C)$, where $\bar{C}(\Sigma_x C) \subset T_x$ denotes the closed unit ball at the origin.*

Proof of Lemma 4.17. We will prove the lemma by induction on n . Clearly when $n = 1$ the lemma holds. Suppose $n - 1$ the lemma holds.

First apply the inductive assumptions to $\Sigma_p \Sigma$ and $\Sigma_p \Sigma_0$, for $p \in \Sigma_0$, we can see that Σ_0 is a topological manifold. Then we have that $\Sigma - \Sigma_0$ has two components H_1, H_2 , each is with set boundary Σ_0 . Observe that for every $x, y \in \bar{H}_i$, $[xy] \subset \bar{H}_i$, if not, by the convexity of Σ_0 , we can get a contradiction. I.e., \bar{H}_i are convex.

By Theorem 2.13, we have that $\partial \bar{H}_i = \Sigma_0, i = 1, 2$, and by Lemma 4.18, we have that $(\bar{H}_i, \partial \bar{H}_i) \xrightarrow{\text{homeo}} (\bar{C}(\Sigma_v \Sigma), \Sigma_v \Sigma)$, where $v \in \bar{H}_i$ is the point such that $|v \partial \bar{H}_i| = \max\{\text{dist}_{\partial \bar{H}_i}\}$. Hence $\Sigma_0 \xrightarrow{\text{homeo}} \Sigma_v \Sigma \xrightarrow{\text{homeo}} S^{n-1}$, where the last one is by the topologically nice property of Σ . The lemma thus follows. \square

Lemma 4.19. *Let $C \in \text{Alex}^n(1)$ with $\partial C = \emptyset$. Let $C_0 \subset C$ be a convex closed subset with dimension $n - 1$ and without boundary. Suppose that C_0 separates C . If there is $v \in C$ such that $|v C_0| \geq \frac{\pi}{2}$, then $[v C_0] \xrightarrow{\text{isom}} S^+(C_0)$, where $[v C_0]$ is*

with the restricted metric. In particular, for $x \in C$ with $|xv| < \frac{\pi}{2}$, we have that $x \in [vv_0]$, for some $v_0 \in C_0$.

Proof. Suppose that $C - C_0$ has two components H_1, H_2 and $v \in H_1$. \bar{H}_1 is convex, as can be seen in the proof of Lemma 4.17. By Theorem 2.13, $\partial\bar{H}_1 = C_0$, and by Lemma 1.2, $|vx| = \frac{\pi}{2}$, for any $x \in C_0$. Then $D(\bar{H}_1) = S(C'_0)$. By the structure of $D(\bar{H}_1)$, C_0 separates $D(\bar{H}_1)$, and thus $C'_0 = C_0$. For $x \in C$ with $|xv| < \frac{\pi}{2}$, we have that $x \in H_1$. The result thus follows. \square

Proof of Theorem 4.6. First by Lemma 4.7 and Remark 4.8, we get that $\dim(\Sigma_1) \leq 1$. And by Sublemma 4.9, we have that Σ_1 is convex and isometric to one of the following: $S^1(r)$ with $r \leq 1$, $[ab]$, $\{v_1, v_2\}$ with $|v_1v_2| = \pi$, $\{v\}$.

We proceed the proof by induction. It is easy to see that for $n = 2$ the theorem holds. Suppose $n - 1$ the theorem holds.

In order to use induction, first we will prove the following:

Sublemma 4.20. *For every $p \in \Sigma_0$ and every $w \in \Sigma_p\Sigma$, we have that $|w\Sigma_p\Sigma_0| \leq \frac{\pi}{2}$ and $|w(\Sigma_p\Sigma_0)^\perp| \leq \frac{\pi}{2}$.*

Proof. Since $\Sigma_p\Sigma_0$ is convex without boundary in $\Sigma_p\Sigma$, by Lemma 1.2 we get that $|w\Sigma_p\Sigma_0| \leq \frac{\pi}{2}$. If $w \in \Sigma_p\Sigma$ such that $|w(\Sigma_p\Sigma_0)^\perp| > \frac{\pi}{2}$, then $|w \uparrow_p^{\Sigma_1}| > \frac{\pi}{2}$. Hence there exists \uparrow_p^q such that $|\uparrow_p^q \uparrow_p^{\Sigma_1}| > \frac{\pi}{2}$, and therefore $d_p \text{dist}_{\Sigma_1}(\uparrow_p^q) > 0$. Thus there exists $y \in \Sigma$ such that $|y\Sigma_1| > \frac{\pi}{2}$, a contradiction to the assumption of the theorem. \square

We will prove the inductive steps according to the four situations of Σ_1 .

Case 1. Assume $\Sigma_1 = S^1(r)$ with $r \leq 1$. By Lemma 4.15, we can get that Σ satisfies the conditions of Theorem 4.4. Then the theorem holds.

Case 2. Assume $\Sigma_1 = \{v_1, v_2\}$ with $|v_1v_2| = \pi$. Then $\Sigma = \Sigma_1 * \Sigma_1^\perp$. Since Σ is topologically nice, Σ_1^\perp is topologically nice and homeomorphic to a sphere. Then by Lemma 4.17, Σ_0 is homeomorphic to S^{n-2} and topologically nice.

Case 3. Assume $\Sigma_1 = \{v\}$. Subcase 1. Assume $\dim(\hat{\Sigma}_1) = n$. For any $x \in \hat{\Sigma}_1$, on one hand, by the assumptions of the theorem, $|xv| \leq \frac{\pi}{2}$, on the other hand, by the definition of $\hat{\Sigma}_1$, $|xv| \geq \frac{\pi}{2}$. Thus $|xv| = \frac{\pi}{2}$. By Lemma 4.7, we get a contradiction.

Subcase 2. Assume $\dim(\hat{\Sigma}_1) = n - 1$. Subsubcase 1. Assume $\partial\hat{\Sigma}_1 = \emptyset$. On one hand, by Lemma 4.17, Σ is homeomorphic to $S(\hat{\Sigma}_1)$, thus $\hat{\Sigma}_1$ separates Σ . On the other hand, for every $w \in \Sigma - (\hat{\Sigma}_1 \cup \Sigma_1)$, we have that $|w\hat{\Sigma}_1| < \frac{\pi}{2}$ and $|w\Sigma_1| < \frac{\pi}{2}$, then by triangle comparison, we get that $\text{dist}_{\hat{\Sigma}_1}$ is noncritical in $\Sigma - (\hat{\Sigma}_1 \cup \Sigma_1)$, thus there is a deformation retraction from $\Sigma - \hat{\Sigma}_1$ to $\{v\}$, a contradiction.

Subsubcase 2. Assume $\partial\hat{\Sigma}_1 \neq \emptyset$. We first show that for any $x \in (\hat{\Sigma}_1)^\circ = (\hat{\Sigma}_1 - \partial\hat{\Sigma}_1)$, $\#\{\uparrow_x^v\} = 2$. By Lemma 4.17, $\Sigma_x\Sigma \stackrel{\text{homeo}}{\simeq} S(\Sigma_x\hat{\Sigma}_1)$, and by the proof of Lemma 4.19, we have that $\#\{\uparrow_x^v\} \leq 2$. Suppose $\#\{\uparrow_x^v\} = 1$, then by Lemma

4.19, there is $w \in \Sigma_x \Sigma$, such that $d_x \text{dist}_v(w) > 0$, a contradiction, as can be seen in the proof of Sublemma 4.20.

By Lemma 4.16, we have that $\Sigma_0 \cap \partial \hat{\Sigma}_1 \neq \emptyset$. If $\Sigma_0 \cap \partial \hat{\Sigma}_1$ is not equal to Σ_0 , i.e., there is $z \in \Sigma_0 \cap (\hat{\Sigma}_1)^\circ$, then by the above paragraph, we have that $\#\{\uparrow_z^v\} = 2$. Thus for any $x \in \Sigma_0$, $\#\{\uparrow_x^v\} = 2$, because by induction and the proof of Sublemma 4.11, we have that for any $p, q \in \Sigma_0$, $\#\{\uparrow_p^v\} = \#\{\uparrow_q^v\}$. By the proof of Sublemma 4.12, we can see that, $\psi : \uparrow_v^{\hat{\Sigma}_1^\circ} \rightarrow \hat{\Sigma}_1^\circ$, $\psi(\uparrow_v^y) = y, y \in \hat{\Sigma}_1^\circ$, is a locally isometric covering map. Since $\hat{\Sigma}_1$ is contractible (the reason is that $\hat{\Sigma}_1 \in \text{Alex}^{n-1}(1)$, with nonempty boundary, hence soul of $\hat{\Sigma}_1$ is a point (cf. [24])), $\uparrow_v^{\hat{\Sigma}_1^\circ} \subset \Sigma_v \Sigma$ is two copies of $\hat{\Sigma}_1^\circ$, by taking closure we can see that $\uparrow_v^{\Sigma_0}$ are two copies of Σ_0 with empty intersection. This is impossible, since by applying the inductive assumptions to $\Sigma_p \Sigma$ and $\Sigma_p \Sigma_0$, for $p \in \Sigma_0$, we can see that Σ_0 is a topological manifold, and one component of $\uparrow_v^{\Sigma_0}$ separates Σ_v , which is homeomorphic to a sphere, into two components, with the closure of each one convex. By Lemma 4.16, we can get a contradiction.

It follows that for any $p \in \Sigma_0$, $\#\{\uparrow_p^v\} = 1$. Hence $\Sigma_0 \subset \partial \hat{\Sigma}_1$. We claim that $\partial \hat{\Sigma}_1 \stackrel{\text{homeo}}{\simeq} S^{n-1}$. Therefore $\Sigma_0 = \partial \hat{\Sigma}_1$. It follows that Σ contains two copies of $S^+(\hat{\Sigma}_1)$, with each copy convex gluing along the boundary, which is homeomorphic to S^n . Thus $\Sigma = [v\hat{\Sigma}_1]$, a double of $v * \hat{\Sigma}_1$.

We now verify the claim: let $s \in \hat{\Sigma}_1$ be a point such that $|s\partial \hat{\Sigma}_1| = \max\{\text{dist}_{\partial \hat{\Sigma}_1}\}$. By Lemma 4.17, $\Sigma_s \Sigma \stackrel{\text{homeo}}{\simeq} S(\Sigma_s \hat{\Sigma}_1)$ and $\Sigma_s \hat{\Sigma}_1 \stackrel{\text{homeo}}{\simeq} S^{n-1}$. By Lemma 4.18, $(\hat{\Sigma}_1, \partial \hat{\Sigma}_1) \stackrel{\text{homeo}}{\simeq} (\bar{C}(\Sigma_s \hat{\Sigma}_1), \Sigma_s \hat{\Sigma}_1)$. Hence $\partial \hat{\Sigma}_1 \stackrel{\text{homeo}}{\simeq} S^{n-1}$.

Subcase 3. Assume $\dim(\hat{\Sigma}_1) = n - 2$. By Lemma 4.14, we get that $\hat{\Sigma}_1 = \Sigma_0$. Hence for every $w \in \Sigma - (\Sigma_0 \cup \Sigma_1)$, $|w\Sigma_0| < \frac{\pi}{2}, |w\Sigma_1| < \frac{\pi}{2}$. By triangle comparison, we get that dist_{Σ_i} is noncritical in $\Sigma - (\Sigma_0 \cup \Sigma_1)$. Therefore there are deformation retractions from $\Sigma - \Sigma_1$ to Σ_0 , and $\Sigma - \Sigma_0$ to Σ_1 . Since for $p \in \Sigma_0$, by induction, $\Sigma_p \Sigma_0$ is homeomorphic to a sphere, and thus Σ_0 is a topological manifold. As in Theorem 4.4, by using Alexander duality, we get a contradiction.

Case 4. Assume $\Sigma_1 = [ab]$. Subcase 1. Assume $\dim(\hat{\Sigma}_1) = n$. Similarly as Subcase 1 of Case 3, we can get a contradiction.

Subcase 2. Assume $\dim(\hat{\Sigma}_1) = n - 1$. First by Lemma 4.7, we have that $\partial \hat{\Sigma}_1 \neq \emptyset$. By Remark 1.3, we have that if there exist $x \in [ab]^\circ$ and $y \in (\hat{\Sigma}_1)^\circ$, such that $|xy| = \frac{\pi}{2}$, then for every $\bar{x} \in [ab]^\circ$ and every $\bar{y} \in \hat{\Sigma}_1$, $|\bar{x}\bar{y}| = \frac{\pi}{2}$. By Lemma 4.7, we get a contradiction. Hence for every $x \in [ab]^\circ$ and every $y \in (\hat{\Sigma}_1)^\circ$, $|xy| > \frac{\pi}{2}$. Since by Lemma 1.2, we have that for every $x \in [ab]$ and for every $z \in \Sigma_0$, $|xz| = \frac{\pi}{2}$. It follows that $\Sigma_0 \subset \partial \hat{\Sigma}_1$.

Let $s \in \hat{\Sigma}_1$ be the point such that $|s\partial \hat{\Sigma}_1| = \max\{\text{dist}_{\partial \hat{\Sigma}_1}\}$. By Lemma 4.17, $\Sigma_s \Sigma \stackrel{\text{homeo}}{\simeq} S(\Sigma_s \hat{\Sigma}_1)$, and $\Sigma_s \hat{\Sigma}_1 \stackrel{\text{homeo}}{\simeq} S^{n-2}$. And by Lemma 4.18, $(\hat{\Sigma}_1, \partial \hat{\Sigma}_1) \stackrel{\text{homeo}}{\simeq} (\bar{C}(\Sigma_s \hat{\Sigma}_1), \Sigma_s \hat{\Sigma}_1)$. Hence $\partial \hat{\Sigma}_1 \stackrel{\text{homeo}}{\simeq} S^{n-2}$. On the other hand, by applying the

inductive assumptions to $\Sigma_p \Sigma$ and $\Sigma_p \Sigma_0$, for $p \in \Sigma_0$, we can see that Σ_0 is a topological manifold. Hence $\Sigma_0 = \partial \hat{\Sigma}_1$.

Observe that for $z \in \Sigma_0$, by Lemma 1.2 and 2.6, we have that $\uparrow_z^{\Sigma_1^\circ} \stackrel{\text{homeo}}{\cong} (a, b) \times \uparrow_z^x$, where $x \in [ab]^\circ$. Since $\uparrow_z^{\Sigma_1} \subset (\Sigma_z \Sigma_0)^\perp$, whose dimension (by Lemma 4.7) ≤ 1 , we have that $\#\{[xz]\}$ is finite. By the proof of Sublemma 4.12, we can see that, $\psi_0 : \uparrow_x^{\Sigma_0} \rightarrow \Sigma_0$, $\psi_0(\uparrow_x^y) = y, y \in \Sigma_0$, is a locally isometric covering map. Since $\Sigma_0 \stackrel{\text{homeo}}{\simeq} S^{n-2}$, $\uparrow_x^{\Sigma_0}$ are several copies of Σ_0 . On the other hand, suppose that $\Sigma_x \Sigma = S(\Sigma')$, then Σ' is homeomorphic to S^{n-2} (by the topologically nice property of Σ). Hence $\Sigma' = \uparrow_x^{\Sigma_0} \stackrel{\text{isom}}{\cong} \Sigma_0$, and $\#\{[xz]\} = 1$. By now we get that $[\Sigma_0 \Sigma_1] = \Sigma_0 * \Sigma_1$, which can be seen in the final part of the proof of Theorem 4.4.

Finally, we need to show that for every $x \in (\hat{\Sigma}_1)^\circ$, $|ax| = \frac{\pi}{2}$ and $|bx| = \frac{\pi}{2}$. Since by the construction of $\hat{\Sigma}_1$ and by the condition of the theorem, $|x, [ab]| = \frac{\pi}{2}$, and we have showed that $|x, [ab]^\circ| > \frac{\pi}{2}$, thus $|ax| = \frac{\pi}{2}$ or $|bx| = \frac{\pi}{2}$, without loss of generality, suppose that $|ax| = \frac{\pi}{2}$. We have that $\#\{[ax]\} = 1$. If not, as Subsubcase 2 in Case 3, we get that $\Sigma = D(S^+(\hat{\Sigma}_1))$, a contradiction. Then by Lemma 4.17 and Lemma 4.19, $|\uparrow_x^a \uparrow_x^b| > \frac{\pi}{2}$. It follows that $d_x \text{dist}_{\Sigma_1}(\uparrow_x^b) > 0$. This contradicts to the condition of the theorem, as can be seen in the proof of Sublemma 4.20.

Thus Σ contains $[\Sigma_0 \Sigma_1]$ and two copies of $S^+(\hat{\Sigma}_1)$, gluing along two copies of $S^+(\Sigma_0)$, which is homeomorphic to S^n . Hence $\Sigma = (a * \hat{\Sigma}_1 \cup_{\hat{\Sigma}_1} b * \hat{\Sigma}_1) \cup_{\partial} (\Sigma_1 * \Sigma_0)$.

Subcase 3. Assume $\dim(\hat{\Sigma}_1) = n - 2$. Similarly as the proof of Subcase 3 of Case 3, we get a contradiction. \square

Now we are ready to verify Proposition 2.1.

Proof of Proposition 2.1. Observe that for any $p \in S$, we have that $\uparrow_p^{\partial \Omega_c} \subset (\Sigma_p S)^\perp$, and by applying first variation formula to the Busemann function, we get that for all $v \in \Sigma_p A$, $|v \uparrow_p^{\partial \Omega_c}| \leq \frac{\pi}{2}$. Hence $|v(\Sigma_p S)^\perp| \leq \frac{\pi}{2}$. Thus for $p \in A$, $\Sigma_p A$ satisfies the conditions of Theorem 4.6. Therefore (2.1.1), (2.1.2) and (2.1.3) can all be derived from Theorem 4.6.

(2.1.4): As can be seen from the proof of Theorem 4.6 that for any $x \in [ab]^\circ$, there is $y \in \hat{\Sigma}_1^p$ such that $|xy| > \frac{\pi}{2}$. Then either a or b must be in E . If $b \notin E$, then also from the proof, there exists a point with distance bigger than $\frac{\pi}{2}$ to a . The first statement thus follows. The second statement is easy to be seen. \square

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